CS 3313:
Foundations of Computing

Review of
Proof Techniques

Outline

- Math preliminaries
- Proof techniques

Sets and set operations

- Set: Collection of non-repeating items $S = \{1, a, \{c, d\}\}, |S| = 3$
- Common sets:

  $\mathbb{Z}$ - set of integers $\qquad$ $\mathbb{Z}^+$ - set of positive integers

  $\mathbb{N}$ - natural numbers $\qquad$ $\mathbb{R}$ - Reals

  $\emptyset$ or $\{\}$ – empty set
- Set relations:
  - Membership: $5 \in \mathbb{Z}$, $3.1 \notin \mathbb{Z}$, $\{c, d\} \in \{1, a, \{c, d\}\}$
  - Subset: $\{1,2\} \subset \{1,2,3\}$
  - Union: $A \cup B$ $\qquad$ Intersection: $A \cap B$ $\qquad$ Complement: $\overline{A}$
  - De Morgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$, $\quad \overline{A \cap B} = \overline{A} \cup \overline{B}$
  - Cartesian product: If $A = \{1,2,3\}, B = \{a, b\}$
    $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

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Proofs

This class will involve a lot of proofs.

General proof procedure:

1. Understand the statement – without the math lingo
2. Build up an intuition – in English or by picture, work through examples
   - This gets easier with practice
3. Construct proof
   - This part is procedural
   - Use facts and theorems you already know
   - Proof techniques will guide you

Writing proofs

- Be concise – no multi-paragraph explanations
- Be precise – use mathematic notation and logical reasoning
- Follow proof techniques – this will give you a structure for the proof

Proof techniques

- Direct proof
- Proof by contradiction
- Proof by induction

Direct Proof

- Produce a chain of logically sound deductions that justify the expected conclusion

Example

- Theorem: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

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- Proof: Important, there are two directions to prove!
  1. Suppose $x \in \overline{A \cup B}$, $then\ x \in \overline{A} \cap \overline{B}$
     If $x \in \overline{A \cup B}$, $then\ x \notin A \cup B$ *(by definition of complement)*
     So, $x \notin A$ and $x \notin B$ (by definition of union)
     Thus, $x \in \overline{A}$ and $x \in \overline{B}$ (by definition of complement)
     Therefore, $x \in \overline{A} \cap \overline{B}$ *(by definition of intersection)*

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     So, $x \notin A$ and $x \notin B$ (by definition of union)
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     Therefore, $x \in \overline{A} \cap \overline{B}$ *(by definition of intersection)*
  2. Suppose $x \in \overline{A} \cap \overline{B}$, $then\ x \in \overline{A \cup B}$
     If $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}\ and\ x \in \overline{B}$ (by definition of intersection)
     So, $x \notin A$ and $x \notin B$ (by definition of complement)
     thus, $x \notin A \cup B$ (by definition of union)
     implying that $x \in \overline{A \cup B}$ (by definition of complement)

Proof by contradiction

Proof Outline:

1. Assume the opposite of what you want to try to prove
2. Show that it leads to a contradiction
3. Thus, the original assumption must be false

Example

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- Intuition: The product of two odd numbers has no factors of 2, so it will be odd
- Proof:
  1. Assume there exists an even n s.t. n^2 is odd (very important to get the negation of the statement correct!)
     Then, $n=2m$ for some integer m (by definition of even)
     So, $n^2 = 4m^2 = 2(2m^2)$ which is even

     Contradiction!!!

Proof by induction

Proof Outline:

1. **Base case**: Verify that statement holds for base case (e.g., true for i=1)
2. **Inductive hypothesis**: Assume that if the statement holds for i=n for some value n
3. **Induction step:** Prove that the statement holds for i=n+1

<u>Why this works:</u>

P(1) is true implies P(2) is true

P(2) is true implies P(3) is true

…

P(n-1) is true implies P(n) is true

Therefore, P(n) is true

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- Proof:

Base case: n = 1 – 1=(1+1)*1/2 = 2/2 = 1

Hypothesis: Assume $\sum_{i=1}^{k} i = \frac{(k+1)k}{2}$ for some k

Induction: Show that $\sum_{i=1}^{k+1} i = \frac{\left((k+1)+1\right)(k+1)}{2}$

- $\sum_{i=1}^{k+1} i = k + 1 + \sum_{i=1}^{k} i = (k+1) + \frac{(k+1)(k)}{2}$ (by hypothesis)
  $= \frac{k^2+3k+2}{2} = \frac{(k+2)(k+1)}{2}$