Foundations of Computing

Lab 4 – PDAs and CFGs

February 12, 2025

Outline

Computing With a PDA

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At each step, a PDA can do the following

1. Read a symbol from the input tape
2. Optionally, pop a value from the Stack
3. Use the input symbol and the stack symbol to choose a next state
4. Optionally, push a value onto the Stack

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Observations:

- Since the control is an NFA, ϵ transitions are allowed
- A PDA may choose not to touch the stack in a particular step
- Unlike the case for DFA/NFA, deterministic PDA's are not equal to non-deterministic ones. We will only study non-deterministic PDAs.

An Example – the Power of Non-determinism

Build a PDA that recognizes the language

$$L = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$$

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- Can similarly check if number of c's matches number of a's
- But, how do we know which one to match?
- Answer: Just guess which one to match non-deterministically, and then verify that guess was correct

Another Example – the Power of Non-determinism

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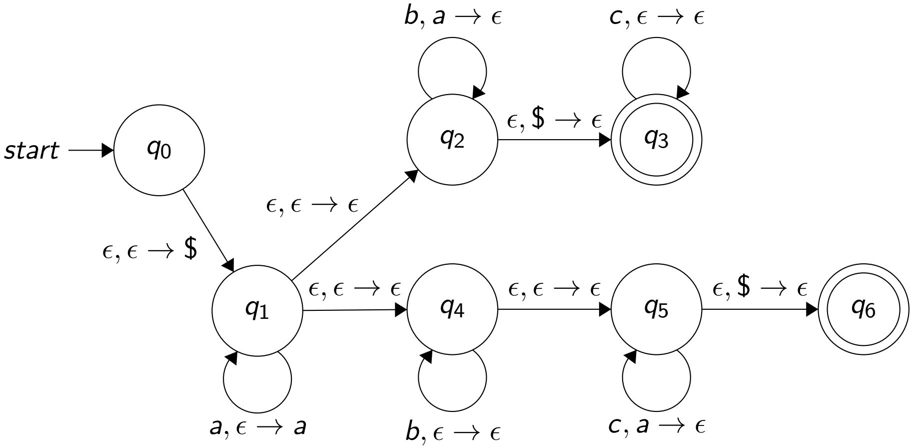

An Exercise – Work in Groups

1. Give a PDA M recognizing

$$L = \{ww^R \mid w \in \{0,1\}^*\}$$

Outline

Grammar

A grammar G consists of:

- V – finite set of variables (usually Capital Letters)
- Σ – a finite set of symbols called the terminals (usually lower case letters)
- R – finite set of rules how strings in L can be produced
- $S \in V$ – start variable

If no S is specified, can assume it is the variable in the first rule.

Definition

For a grammar G, the language L_G generated by G is the set of all terminal strings that can be produced by G starting with the start symbol by using a sequence of the production rules.

Strings Produced by a Grammar

For a grammar G generating language L, can generate each string $w \in L$ as follows:

1. Write down the start variable
2. Find a written-down variable and a rule starting with that variable. Replace the written variable with the right side of that rule
3. Repeat Step 2 until no variables remain

Definition

A grammar G is context-free if for all of its rules, the right side consists of exactly one variable and no terminals.

How to Design CFGs for L

Designing CFGs

- CFGs are inherently recursive (e.g., $A \to 0A1$) – need to think what happens when we recurse
- Build a string from outside in
- Build from both ends at the same time (due to recursion)

This is Tricky

Designing CFGs is not natural, takes lots of practice

Example 1

Question

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- Consider the string aaaaabbccc
  - Red part on the inside
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- Generate outside part first, and then inside part

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  Solution:

$$
\begin{aligned}
S &\rightarrow aSc \mid B \mid \epsilon \\
B &\rightarrow aBb \mid \epsilon
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- Every time we add an a, should also add a b
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Solution:

$$S \rightarrow SaSbS \mid SbSaS \mid \epsilon$$

Exercises

Construct CFGs for the following languages:

2. $\{a^n b^m \mid 2n \leq m \leq 3n\}$
3. $\{w \mid w \in \{a, b\}^* \text{ and } n_a(w) \neq n_b(w)\}$

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