Foundations of Computing

Lecture 12

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February 27, 2024

Outline

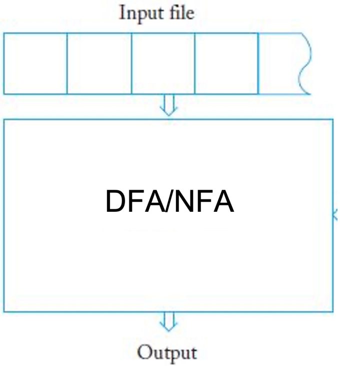
1. Lecture 10+11 Review
2. Models of Computation
3. The Turing Machine
4. Formalizing Turing Machines

Lecture 10+11 Review

- Equivalence of CFGs and PDAs
- CFL Pumping Lemma
- Using the CFL Pumping Lemma

Outline

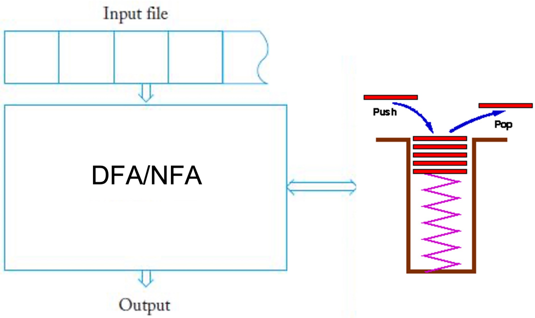
Finite Automata



Recall:

- An NFA/DFA has no external storage
- Only memory must be encoded in the finite number of states
- Can only recognize regular languages

Pushdown Automata (PDA)



A PDA consists of:

- An NFA for a control unit
- A Stack for storage

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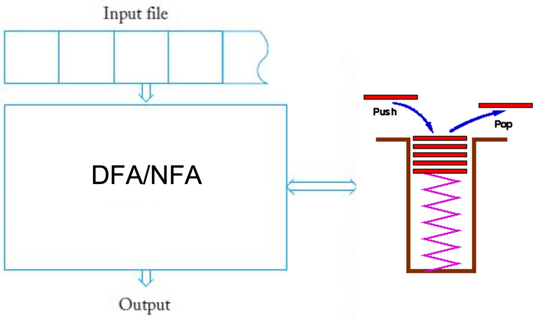

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Recall:

- Can only access memory in LIFO fashion
- Can only recognize context-free languages

A Model for General Computation

Question

All the prior models of computation couldn't recognize some simple languages. Can we develop a computation model that captures all languages that can be computed on any computer?

Our Goal

One model to rule them all!

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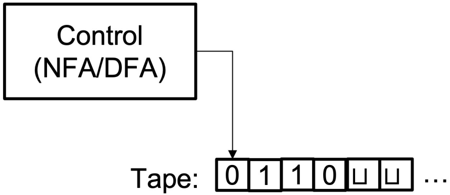

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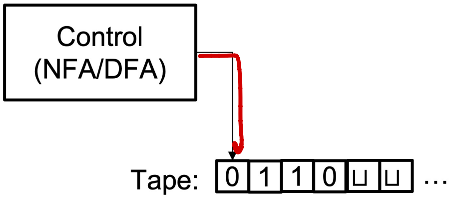

Key Differences:

- A TM can read and write to its tape

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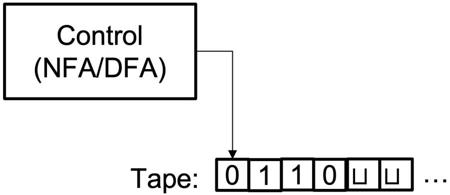

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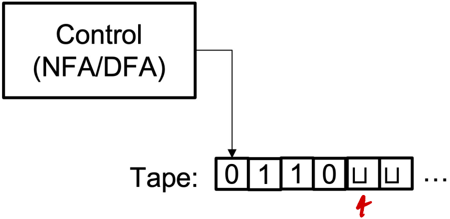

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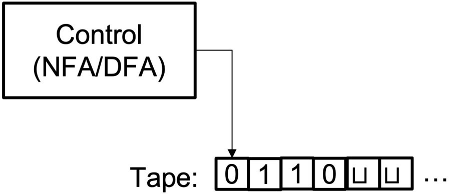

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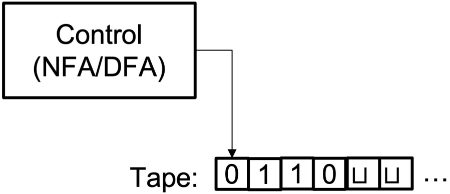

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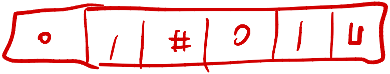
- Control FA has accept and reject states that are immediately output if entered

An Algorithm for M:
On input string s (written on the tape):



An Example: TM To Recognize $L = \{w\#w \mid w \in \{0,1\}^*\}$

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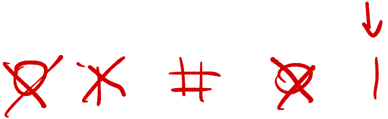

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3. When all symbols to the left of # have been crossed off, check that no uncrossed-off symbols remain to the right of #. If any symbols remain, reject, otherwise accept.

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- To show how to solve a problem, we design an algorithm
- To reason about languages accepted by NFA/PDA, we designed algorithms
- How can we reason about the limits of what an algorithm can compute?

Turing Machines and Algorithms

Church-Turing Thesis (1936)

Anything that can be computed by an algorithm can be computed by a Turing Machine

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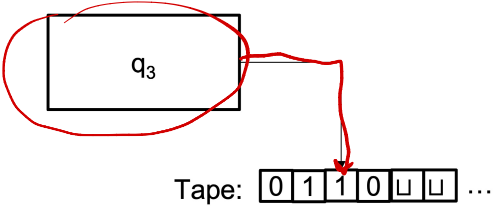
Computing on a Turing Machine



Configuration of a TM

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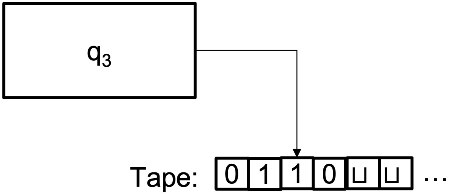

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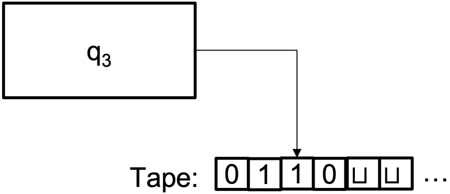

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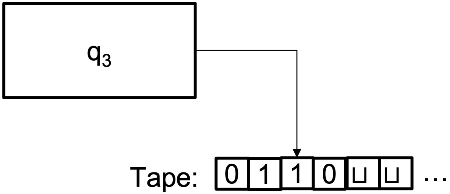

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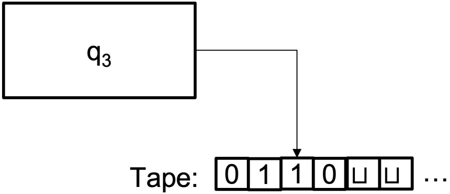

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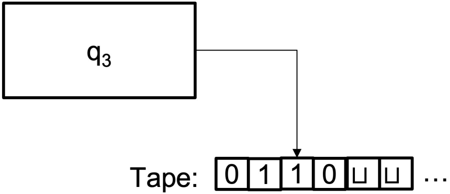

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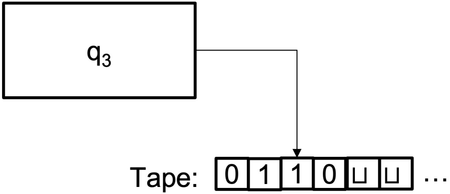

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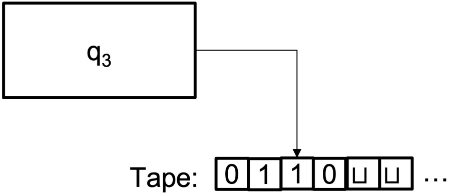

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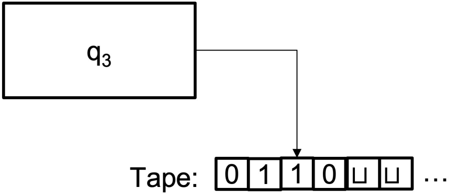

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- halting configuration – accepting or rejecting configs

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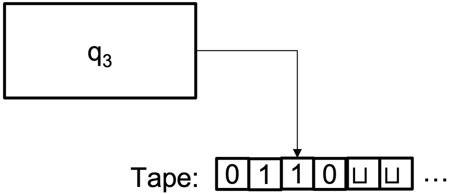

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Language $L(M)$

The collection of strings that M accepts

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Definition: Decidable languages

A language L is *decidable* or *recursive* if some TM M decides it

- M halts on all inputs, accepting those in L and rejecting those not in L

Another Example

Consider $L = \{0^{2^n} \mid n \geq 0\}$

TM algorithm M for recognizing L:
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5. Go to step 1

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On input s:

1. If the tape has exactly one 0, accept
2. If the tape has an odd number of 0's, greater than 1, reject
3. Sweep left to right across tape, crossing out every other 0
4. Return the head to the left-hand end of the tape
5. Go to step 1

Making M Formal

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_a, q_r\}$
- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \sqcup\}$
- δ :

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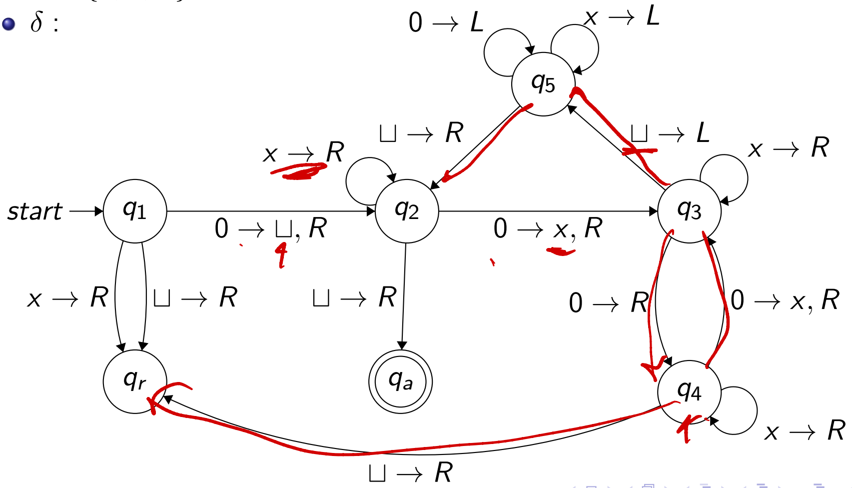

Running M on $w = 0000$