Foundations of Computing

Lecture 13

Arkady Yerukhimovich

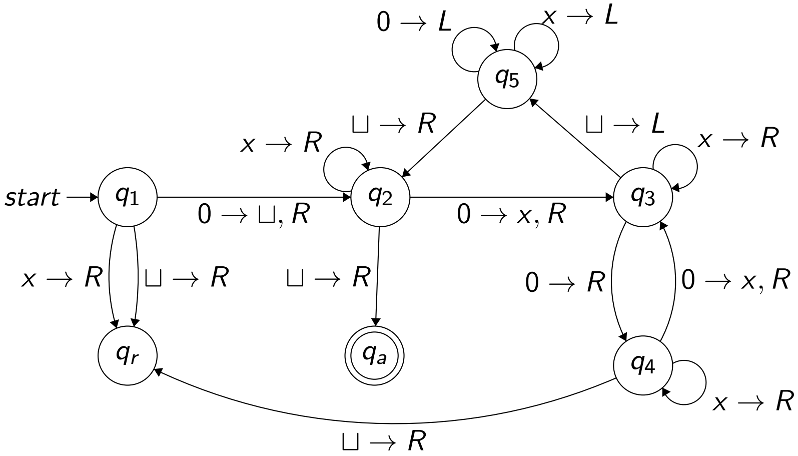
February 29, 2024

Outline

Lecture 12 Review

- Turing Machines
  - Definition
  - Examples
- Church-Turing Thesis
  Informally: Anything that can be computed can be computed by a Turing Machine.

Running M on $w = 00$



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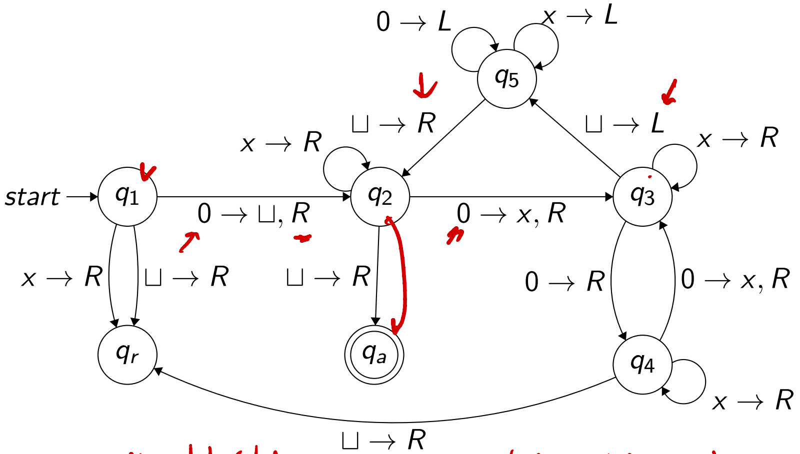

$\rightarrow q_5 \sqcup x \sqcup \rightarrow \sqcup q_2 x \sqcup \rightarrow \sqcup x q_2 \sqcup \rightarrow q_a$

Let's consider an execution on input 00 (as a sequence of configurations)

$q_1 0 0 \sqcup \rightarrow \sqcup q_2 0 \sqcup \rightarrow \sqcup x q_3 \sqcup \rightarrow \sqcup q_5 x \sqcup$

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Specification of a Turing Machine

There are several levels of detail for specifying a TM

1. Full specification
   - Give full detail of transition function δ
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    - Explain algorithmically what happens on the tape
    - For example, scan the tape until you find a $\#$, zig-zag on the tape, etc.
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   - Explain algorithmically what happens on the tape
   - For example, scan the tape until you find a $\#$, zig-zag on the tape, etc.
   - Don't bother specifying a DFA for the control state
3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

Example 1: $L = \{a^i b^j c^k \mid i \times j = k \text{ and } i, j, k \geq 1\}$

Machine M deciding L

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4. Restore all the b's, find next uncrossed off a and repeat Step 3.
5. If all a's are crossed off, check if all c's are crossed off. Accept if yes, reject if no.

Example 2 – Build a TM deciding L Below

$$L = \{\#x_1\#x_2\#\cdots\#x_\ell \mid \text{ each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for all } i \neq j\}$$

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2. Place mark on top of first $\#$ and scan to next $\#$ and mark it. If no second $\#$ found, accept.

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4. Move right mark to next #, if there isn't one move left mark one # to the right and right mark to # after that (if there isn't one, accept)

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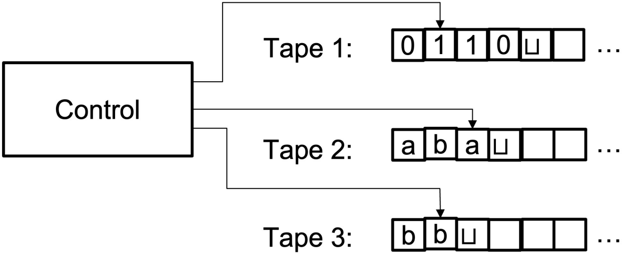
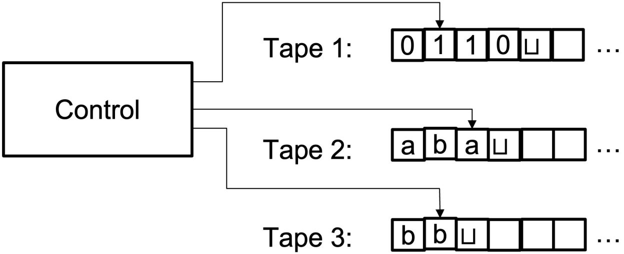
5. Goto step 3

Outline

Multi-Tape Turing Machines



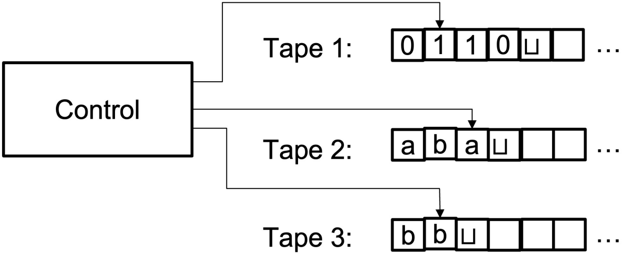
Multi-Tape Turing Machines



In each step:

- M can read each tape
- M can write to each tape
- M can move each tape head Left or Right

Multi-Tape Turing Machines



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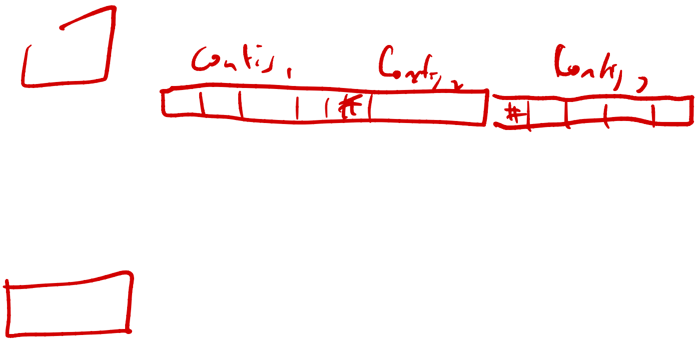
Formally, for k tapes

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$$

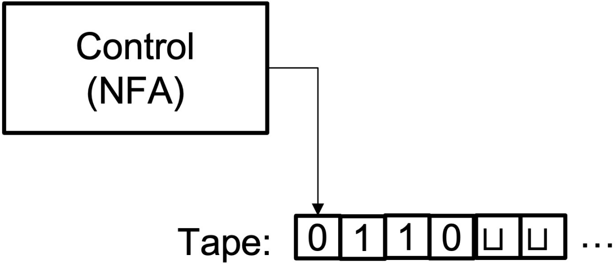
Multi-Tape Turing Machines

Theorem

Every multi-tape TM has an equivalent single-tape TM



Nondeterministic Turing Machines



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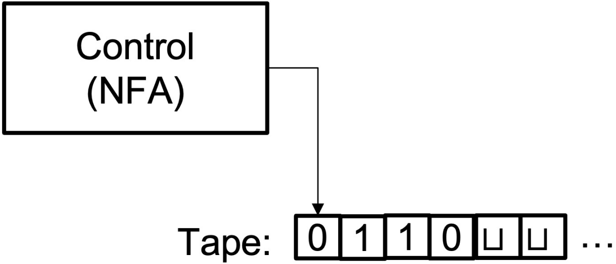

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$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

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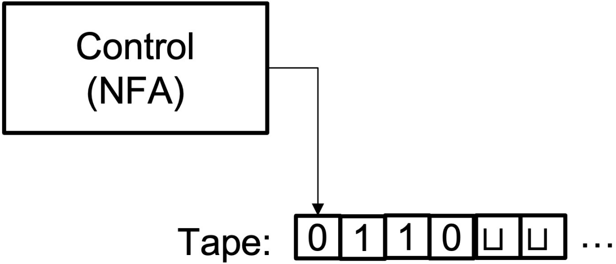

Formally,

$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

Intuition:

- The control unit is non-deterministic - many transitions possible on each input
- Execution corresponds to a tree of possible executions
- Accept if any of possible execution leads to accept

Nondeterministic Turing Machine

Theorem

Every nondeterministic TM has an equivalent deterministic TM.

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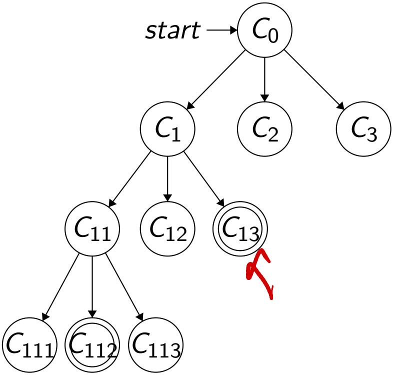

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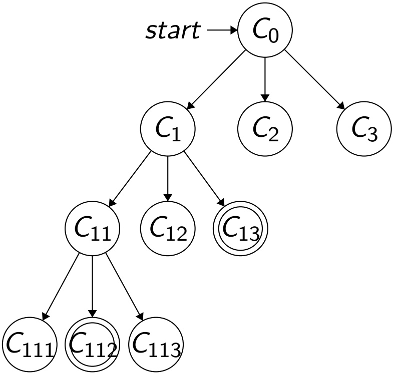

- Recall that an execution of a DTM is a sequence of configurations

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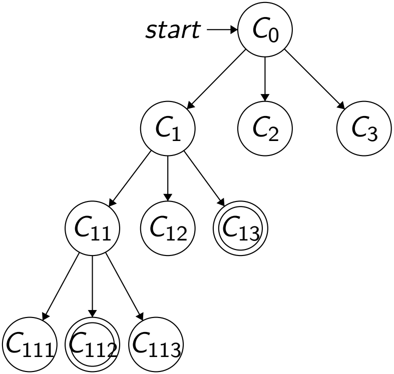

- Recall that an execution of a DTM is a sequence of configurations
- Execution of an NTM is a tree of configurations (branches correspond to non-deterministic choices)

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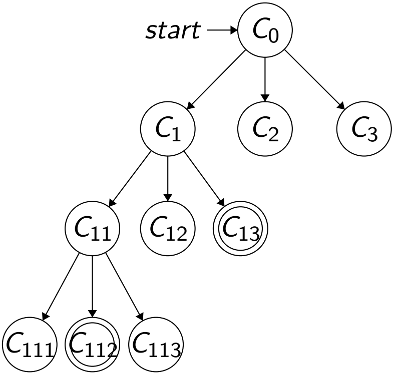

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- If any node in the tree is an accept node, the NTM accepts

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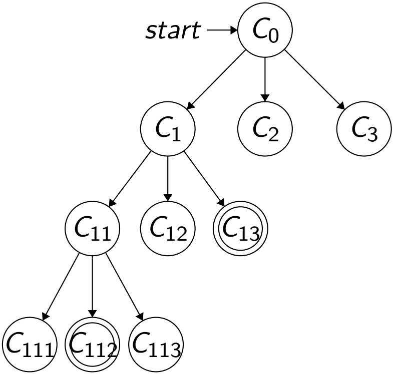
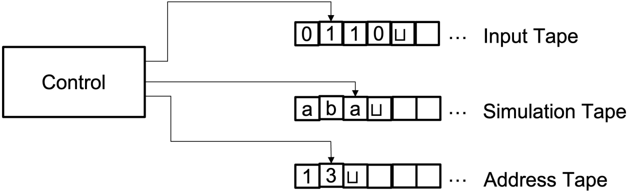
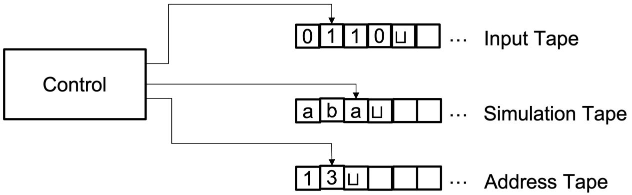


- Recall that an execution of a DTM is a sequence of configurations
- Execution of an NTM is a tree of configurations (branches correspond to non-deterministic choices)
- If any node in the tree is an accept node, the NTM accepts
- To simulate an NTM by a DTM, need to try all configurations in the tree to see if we find an accepting one



To simulate an NTM N by a DTM D, we use three tapes:

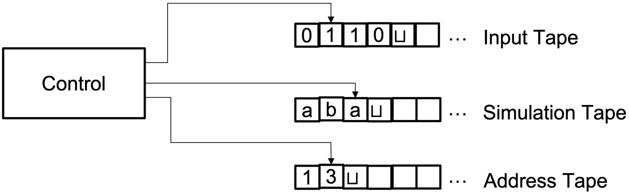
Nondeterministic Turing Machines



To simulate an NTM N by a DTM D, we use three tapes:

1. Input tape – stores the input and doesn't change

Nondeterministic Turing Machines



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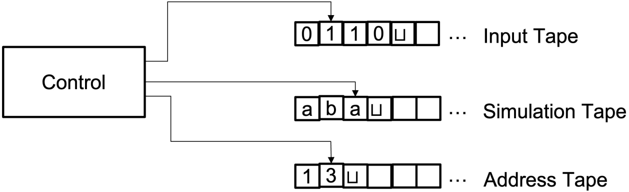
Nondeterministic Turing Machines



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1. Input tape – stores the input and doesn't change
2. Simulation tape – work tape for the NTM on one branch of nondeterminism
3. Address tape – use to store which nondeterministic branch you are on

Nondeterministic Turing Machines

Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty

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2. Copy w to tape 2

Nondeterministic Turing Machines

Simulating an NTM N

1. Start with input w on tape 1, and tapes 2,3 empty
2. Copy w to tape 2
3. Use tape 2 to simulate a run of N. Whenever it needs to make a non-deterministic choice, see next symbol on tape 3 for which branch to take. If no symbols left, go to step 4

Nondeterministic Turing Machines

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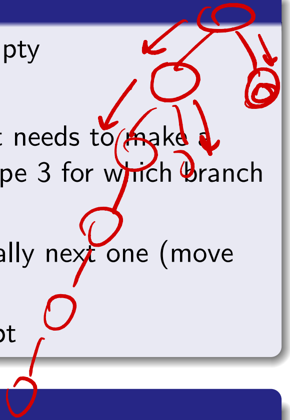

Important

Must traverse NTM tree in breadth-first, not depth-first order

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Important

Must traverse NTM tree in breadth-first, not depth-first order

- Depth-first traversal may get stuck in an infinite loop, and not detect terminating branch

Next Week

- Languages about machines
- Decidable and undecidable languages