Foundations of Computing

Lecture 15

Arkady Yerukhimovich

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Outline

1. Lecture 14 Review
2. Review: Decidable Languages
3. Preliminaries – Countable and Uncountable Sets
4. Proving A_{TM} Undecidable
5. Reductions between Languages

Lecture 14 Review

- Decidable and Turing-recognizable languages
- Decidability of regular and context-free languages

Outline

Characterizing Computability of Languages

Definition: Decidable languages

A language L is *decidable* or *recursive* if some TM M decides it

- M halts on ALL inputs, accepts those in L and rejects those not in L
- Seems to match informal definition we wanted before

Definition: Turing-recognizable languages

A language L is *Turing-recognizable* or *recursively enumerable* if some TM M recognizes it

- M halts and accepts all strings in L
- M may not halt on strings not in L – does not necessarily have to reject

Observation

Every Decidable language is also Turing-recognizable, but the reverse direction is not true.

Decidable Languages

We showed the following languages are decidable:

- Languages about Finite Automata
  1. $A_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
  2. $A_{NFA} = \{\langle B, w\rangle \mid B \text{ is a NFA that accepts input string } w\}$
  3. $A_{REX} = \{\langle R, w\rangle \mid R \text{ is a reg. exp. that generates the string } w\}$
  4. $E_{DFA} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
  5. $EQ_{DFA} = \{\langle A, B\rangle \mid A, B \text{ are DFAs and } L(A) = L(B)\}$
- Languages about CFGs
  1. $A_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$
  2. $E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$

A Language About Turing Machines

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) = 1\}$$

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  1. Simulate M on input w
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- Is A_{TM} Decidable?
  - The problem: M may never halt
  - In this case, above algorithm will never output accept or reject
  - If could determine that M will never halt (i.e, it has entered an infinite loop), could reject.

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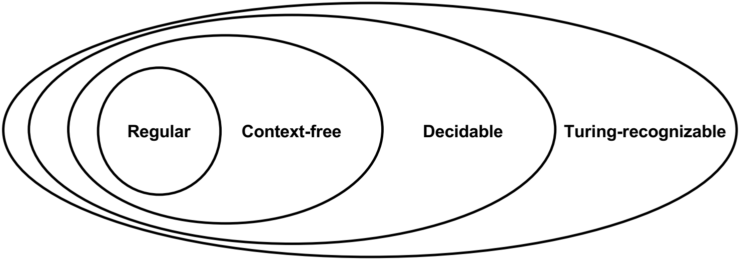
An Undecidable Problem

- We will prove today that A_{TM} is undecidable

Relationships Among Language Classes



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Set Cardinality

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- $|S_1| = |S_2|$ if there's a one-to-one and onto mapping from S_1 to S_2
- Example:
  $A = \{0, 1, 2, 3\}$
  $B = \{a, b, c, d\}$
  $f(0) = a, f(1) = b, f(2) = c, f(3) = d$

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- A set A is countable if it is finite or countably infinite
- A set that is not countable is *uncountable*

Example 1: Evens

Evens

The set of even numbers is

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Evens

The set of even numbers iscountable

$$f(x) = \frac{x}{2}$$

Example 2: Rationals

Rationals

The set of rational numbers is

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Rationals

The set of rational numbers is countable

$\frac{1}{1} \to 1$ $\frac{1}{2} \to 2$ $\frac{1}{?} \to ?$

Example 2: Rationals

Rationals

The set of rational numbers is countable

	1	2	3	$\cdots$
1	1 1/1	3 1/2	6 1/3	
2	2 2/1	5 2/2	2/3	$\cdots$
3	4 3/1	3/2	3/3	
4	4/1	4/2	$\vdots$	

Example 3: Strings

Strings

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Strings

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1 ϵ

2 0   3 1

4 00   5 01   6 10   7 11

Diagonalization

Real Numbers

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Proof: By diagonalization

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- Assume that $\mathcal{R}$ is countable

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- Assume that $\mathcal{R}$ is countable
- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

Diagonalization

Real Numbers

The set of real numbers ($\mathcal{R}$) is uncountable

Proof: By diagonalization

- Assume that $\mathcal{R}$ is countable
- Then there is a one-to-one and onto mapping f from $\mathcal{N}$ to $\mathcal{R}$

n	f(n)
1	1.234...
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2.22

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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$

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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$
- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$

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- We already showed that the set of strings $\{0, 1\}^*$ is countable
- Can similarly show that for any finite alphabet Σ, Σ^* is countable
- But, a TM M can be written as a string $\langle M \rangle \in \Sigma^*$
- Hence, by omitting all strings that are not encodings of valid TMs we get a mapping of TMs to $\mathcal{N}$

The Set of Languages

Languages over alphabet Σ

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   - An infinite binary sequence is an infinite length string of 0's and 1's

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   - B is uncountable

Assume f

$f(x)$

x	$f(x)$
1	0 1 0 0 ...
2	1 1 0 0 0 ...
3	1 1 1 0 0 0 ...
⋮	

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   - Define the characteristic sequence χ_A of language $A \in \mathcal{L}$

$$
\begin{array}{rclccccccccl}
\Sigma^* & = & \{ & \epsilon & 0 & 1 & 00 & 01 & 11 & 000 & \cdots & \} \\
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   - This is a one-to-one and onto mapping from $\mathcal{L}$ to B, so $|\mathcal{L}| = |B|$
3. Therefore, $\mathcal{L}$ is uncountable

Some Languages are not Turing-recognizable

We have proven:

1. The set of Turing Machines is countable
2. The set of languages is uncountable

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Where are we now?

- We have proven that some languages are not Turing-recognizable
- But, we have not given any examples of such a language

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Proof: By contradiction

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- Assume that A_{TM} is decided by a TM H

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$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle \end{cases}$$

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Run D on <D>

D(<D>):

- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

- Contradiction!

How Is This a Diagonalization?

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	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*		*accept*	
M_2	*reject*	*reject*	*reject*	$\ldots$	*accept*	$\ldots$
M_3	*accept*	*accept*	*accept*		*reject*	
$\vdots$		$\vdots$		$\ddots$		
D	*reject*	*accept*	*reject*		?	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??

A Turing-unrecognizable Language

$\overline{A_{TM}}$

The language

$$\overline{A_{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M(w) \neq 1\}$$

is not Turing-recognizable

Outline

Another Way to Prove Undecidability

Reductions Between Problems

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Reductions Between Problems

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Observations:

- Reductions not always symmetrical: $A \leq B$ does not mean $B \leq A$
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Intuition

$A \leq B$ means that:

- problem A is no harder than problem B.
- Equivalently, problem B is no easier than problem A

Main Observation

Suppose that $A \leq B$, then:

Reductions and Undecidability

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- If A is undecidable
- B must also be undecidable

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Reductions and Undecidability

Main Observation

Suppose that $A \leq B$, then:

- If A is undecidable
- B must also be undecidable

Proof: (by contradiction)

- Suppose that B is decidable
- Since $A \leq B$, there exists an algorithm (i.e., a reduction) that uses a solution to B to solve A
- But, this means that A is decidable by running the machine for B as needed by the reduction

Undecidability of HALT

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Proof Sketch:

- We show that $A_{TM} \leq HALT$

Undecidability of HALT

$$HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

Theorem: $HALT$ is undecidable

Proof Sketch:

- We show that $A_{TM} \leq HALT$
- Since we know that A_{TM} is undecidable, this shows that $HALT$ is also undecidable

Undecidability of HALT

$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$$

Theorem: $HALT$ is undecidable

Proof Sketch:

- We show that $A_{TM} \leq HALT$
- Since we know that A_{TM} is undecidable, this shows that $HALT$ is also undecidable

Proof:

Construct algorithm S that decides A_{TM} given a TM R that decides $HALT$

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- If R rejects – $M(w)$ doesn't halt – halt and reject

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- Output whatever M output