Foundations of Computing

Lecture 16

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March 18, 2025

Outline

1. Lecture 15 Review
2. Proofs by Reduction
3. Example Proofs by Reduction

Lecture 15 Review

- Countable and Uncountable Sets
  - Diagonalization
- Proving A_{TM} is Undecidable

Announcements

Homework 6 is out

- Due at 5PM on Monday, March 24
- Early deadline by midnight on Friday, March 21.

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- You will again be allowed 2 pieces of paper with notes
- Exam will cover the following topics:
  - Turing Machines
  - Countable and uncountable sets
  - Decidable, Turing-recognizable Languages
  - Proofs by reduction
  - Everything we cover this week
- CFL Pumping Lemma will not be on the exam

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Another Way to Prove Undecidability

Reductions Between Problems

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- problem A is no harder than problem B.
- Equivalently, problem B is no easier than problem A

Reductions and Undecidability

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Suppose that $A \leq B$, then:

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- But, this means that A is decidable by running the reduction using the decider machine for B.

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- Run $D(\langle M, w\rangle)$
- If D rejects – $M(w)$ doesn't halt – halt and reject
- if D accepts – $M(w)$ halts – Simulate $M(w)$ until it halts, and output whatever M outputs

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- If we can decide whether M' recognizes a regular language or not, can use that to decide whether M accepts w or not

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2. Run D on input $\langle M' \rangle$
   - If $M(w) = 1$, then $M'_{\langle M,w \rangle}$ accepts all $x \in \Sigma^*$ – regular
   - If $M(w) \neq 1$, $M'_{\langle M,w \rangle}$ accepts the language $0^n 1^n$ – not regular
3. Output what D outputs

Other Undecidable Languages – Exercise

$$EMPTY-STRING_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$$

Think about:

- What direction should the reduction go?
- What language should the reduction use?