Foundations of Computing

Lecture 17

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Outline

1. Lecture 16 Review
2. Where Are We Now?
3. Reduction Types
4. A Computational Definition of Information – Kolmogorov Complexity

Lecture 16 Review

- Proofs by reduction
- Undecidable languages
  - $HALT_{TM}$
  - $REGULAR_{TM}$

Exercise

$$EMPTY-STRING_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } M(\epsilon) = 1\}$$

$A_{TM} \leq ES_{TM}$

1. Assume ES_{TM} is decidable - $D(\langle M\rangle)$
2. $R(\langle M, w\rangle)$

if $M'(\epsilon)=1$ $M(w)=1$

if M' doesn't accept ϵ then M doesn't accept w

$M' \{ M'(\epsilon)$:

Run $M(w)$ output what it outputs

$D(\langle M'\rangle)$

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1. To show that a problem is decidable: Give an algorithm that always terminates and outputs the answer
2. To show that a problem is undecidable: Give an algorithm (a reduction) that shows that this problem can be used to solve an undecidable problems

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Recall: $\overline{A_{TM}}$ is Turing-unrecognizable

$\overline{A_{TM}} \leq L$

$\langle M, w \rangle$ accept iff
M does not accept w

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Problem: $\overline{A_{TM}} \leq A_{TM}$ — D

$\underline{R}(\langle M, w \rangle)$

$D(\langle M, w \rangle) = \begin{cases} 1 & \text{if } M(w) = 1 \\ 0 & \text{if } M \text{ doesn't accept } w \end{cases}$

Output the opposite

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Problem: $\overline{A_{TM}} \leq A_{TM}$
but A_{TM} is Turing-recognizable

What About Turing-Unrecognizable Problems?

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Can reductions help us determine if a language is Turing-unrecognizable?

Recall: $\overline{A_{TM}}$ is Turing-unrecognizable

Problem: $\overline{A_{TM}} \leq A_{TM}$
but A_{TM} is Turing-recognizable

Takeaway: General reductions do not work for Turing-unrecognizable languages

Solution

We need to restrict what our reductions can do.

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Language A is mapping reducible to language B ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B$$

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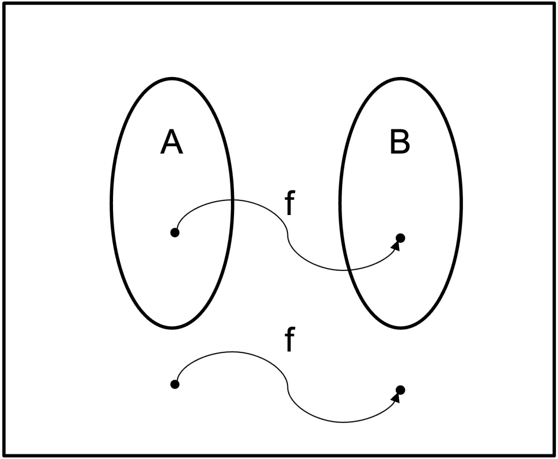
  - Karp reductions (when only considering poly-time reductions)
- Works by mapping input $\in A$ to input $\in B$ and vice-versa

Mapping Reductions



Mapping Reduction Properties

Mapping reductions are very useful:

If $A \leq_m B$

- If B is decidable then A is decidable

$x \in A \Rightarrow f(x) \in B$

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- If A is undecidable then B is undecidable

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- If A is not Turing-recognizable than B is not Turing-recognizable

Observation:

Mapping reductions work for both decidability and Turing-recognizability.

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- For example, in the proof that $L_{TM} \leq L_{E_{TM}}$, we flipped the result of R deciding $L_{E_{TM}}$

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But, they have weaker implications than mapping reductions:

3. If $A \leq_T B$
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   - If A is not decidable, then B is not decidable

   - If B is Turing-recognizable, A is not necessarily Turing-recognizable
   - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

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Information in a String

$$A = 010101010101010101010101$$

$$B = 110100100011100010111111$$

Question

Which of these strings contains more information?

Kolmogorov Complexity

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- $K(x)$ is the minimal description of x
- This captures the "amount of information" in x

Properties of Kolmogorov Complexity

1. $\forall x, K(x) \leq |x| + c$ for some constant c

M - outputs its input

$d(x) = M \| x$

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3. $\forall x, y, K(xy) \leq 2K(x) + K(y) + c$

$$\underbrace{00\,11\,00\,11\,11\,00}_{2K(x)}\,01\,\underbrace{\qquad\qquad\qquad}_{y}$$

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   - So, there exists at least one string that is incompressible
2. In fact, incompressible strings look like random strings
3. But, $K(x)$ is not computable, moreover it is undecidable whether a string is incompressible