Foundations of Computing

Lecture 18 – Exam Review

Arkady Yerukhimovich

March 26, 2024

Outline

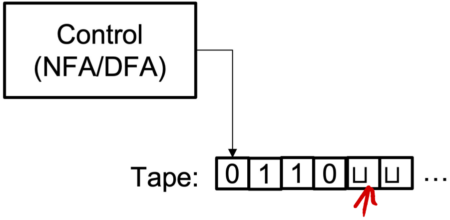
1. Lecture 17 Review
2. Turing Machines
3. Languages Recognized by TMs
4. Undecidable Languages
5. Proofs by Reduction
6. Kolmogorov Complexity

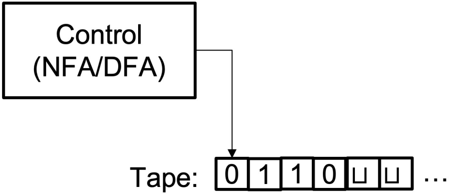
Lecture 17 Review

- Review of Reductions
- Types of Reductions – Mapping reductions, Turing reductions
- A brief intro into Kolmogorov complexity

Outline

The Turing Machine





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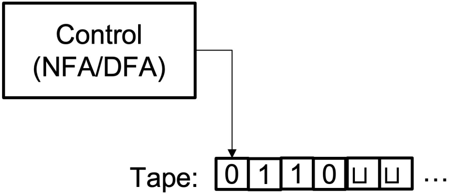

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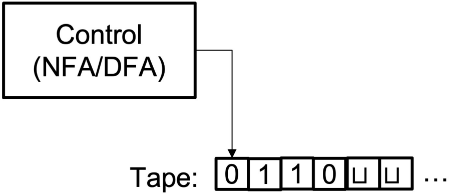

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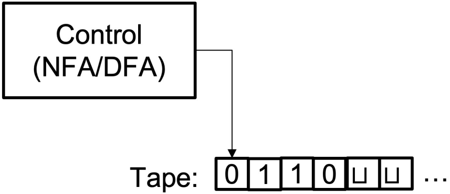

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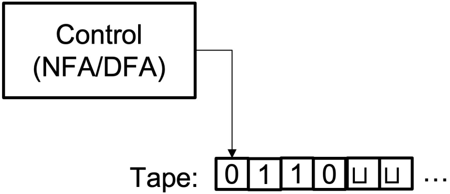

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- No separate input tape, input written onto memory tape at start
- The memory tape is infinite
- Control FA has accept and reject states. If entered, TM halts and outputs.

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3. When all symbols to the left of # have been crossed off, check that no uncrossed-off symbols remain to the right of #. If any symbols remain, reject, otherwise accept.

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- To prove that some problem cannot be solved by an algorithm, enough to reason about Turing Machines
- This means that Turing Machines give an abstraction to capture "feasible computation"

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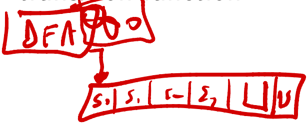

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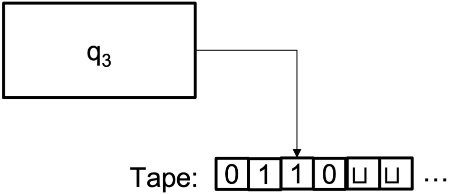

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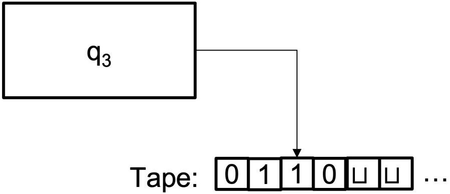

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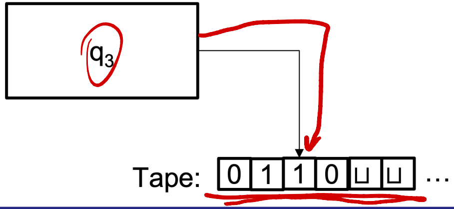

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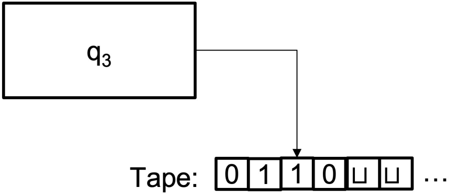

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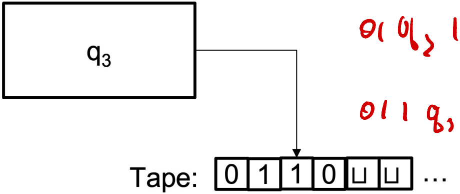

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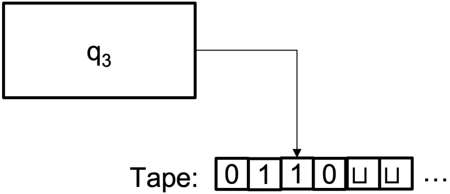

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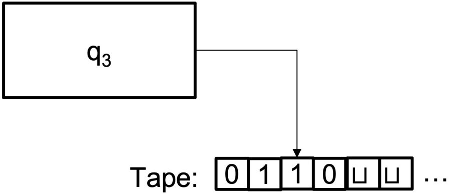

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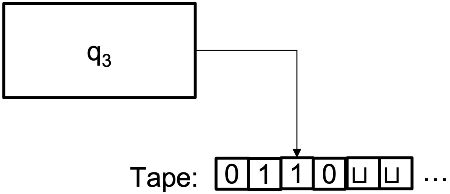

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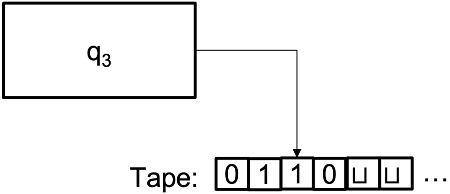

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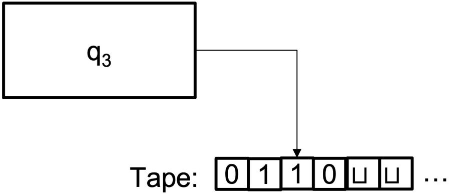

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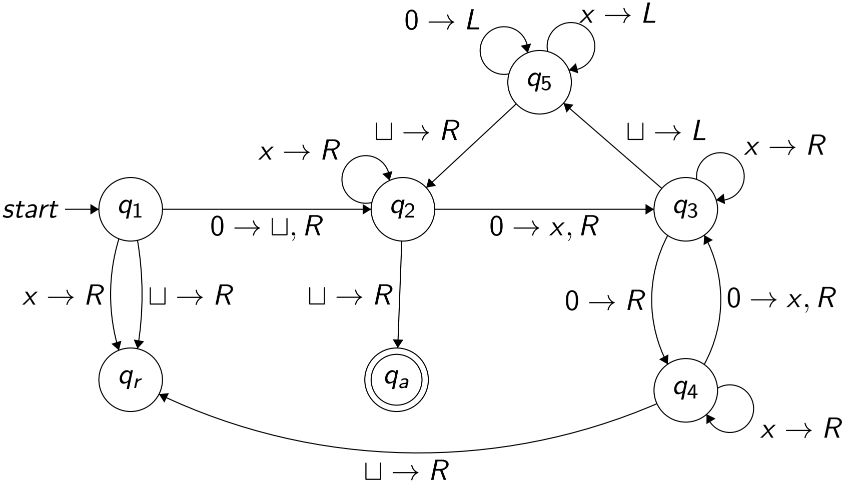
Full Specification: Running M on $w = 0000$



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- M halts on all inputs, accepting those in L and rejecting those not in L

Take Away

You should be able to show that a language is decidable or Turing-recognizable by designing a TM algorithm.

Important TM Notation / Observations

- TM always takes a string as input
  - Sometimes we want to talk about a TM taking another type of input (e.g., a graph, a FA, a TM)
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- Can give a machine as an input to another machine
  - All machines we have seen can be written as finite tuples, e.g. $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$
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  - So, we can write this as a string and pass it to a TM
  - TM can then run the machine from this description
  - A TM that accepts any TM and runs it is called a *universal TM*

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3. Algorithm specification
   - Give algorithm in pseudocode
   - Don't explicitly spell out what happens on the tape

Turing Machine Variants

- Multi-tape Turing Machine
- Nondeterministic Turing Machine

What You Need to Know

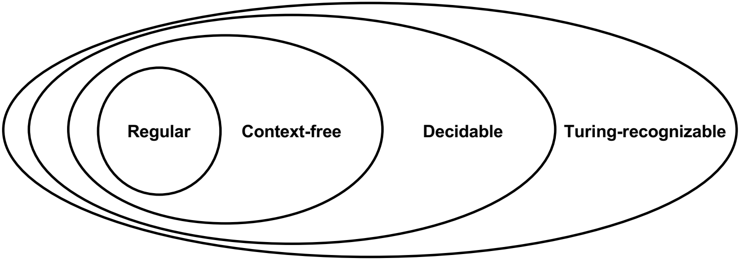
- Be able to explain what the variant is
- Know whether it is equivalent to standard TM
- Be able to explain why

Decidable Languages

We have seen many examples of decidable languages:

- Languages about strings
- Languages about DFAs/NFAs/PDAs/CFGs – know which ones are decidable and which are not, why
- Be comfortable with TM's that take another machine as input

Relationships Among Language Classes



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- A set that is not countable is *uncountable*

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  Idea: For all $i \in \mathcal{N}$, make $x_i \neq f(i)_i$
- Contradiction – f is not mapping between $\mathcal{R}$ and $\mathcal{N}$

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Proof: By construction of machine $M_{A_{TM}}$

$M_{A_{TM}}$: On input $\langle M, w\rangle$,

1. Run M on input w

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Proof: By construction of machine $M_{A_{TM}}$

$M_{A_{TM}}$: On input $\langle M, w \rangle$,

1. Run M on input w
2. If M halts, halt and output what M outputs

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- Note that $M_{A_{TM}}$ may not halt on all inputs – doesn't decide A_{TM}

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- Now consider what happens if we run D on $\langle D\rangle$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

How Is This a Diagonalization?

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*		*accept*	
M_2	*reject*	*reject*	*reject*	$\cdots$	*accept*	$\cdots$
M_3	*accept*	*accept*	*accept*		*reject*	
$\vdots$		$\vdots$		$\ddots$		
D	*reject*	*accept*	*reject*		?	

- We have defined D to do the opposite of what M_i does on input $\langle M_i \rangle$
- But what does D do on input $\langle D \rangle$??

Outline

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Suppose that $A \leq B$, then:

Reductions and Undecidability

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- But, this means that A is decidable by running the machine for B as needed by the reduction

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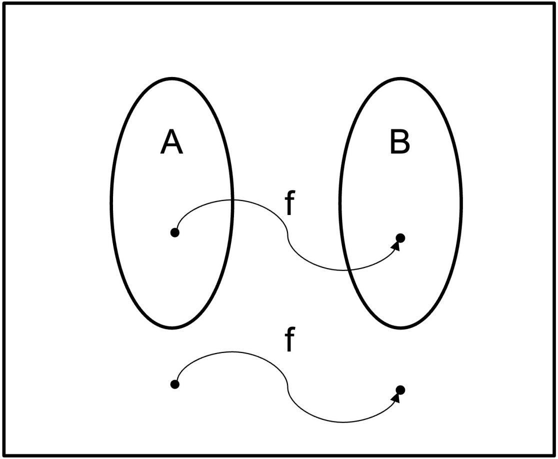
Reduction Types

Know the difference between:

- Mapping reductions
- Turing reductions

Know what each one implies

Mapping Reductions



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Language A is Turing reducible to language B ($A \leq_T B$) if can use a decider for B to decide A.

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- The reduction may make multiple calls to decider for B and may not directly use the result.

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4. If $A \leq_T B$
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   - If A is not Turing-recognizable, cannot say if B is Turing-recognizable

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2. The Kolmogorov complexity of x is

$$K(x) = |d(x)|$$

- $K(x)$ is the minimal description of x
- This captures the "amount of information" in x

What You Need to Know

- Basic definition of Kolmogorov complexity
- Be able to find rough bounds on Kolmogorov complexity
- Don't need to be able to prove anything