Foundations of Computing

Lecture 19

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Question

Suppose we want to solve a problem in real life, is knowing that it is decidable enough?

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Complexity

The study of decidability under bounded models of computation

Outline

1. Polynomial Time
2. The Complexity Class $\mathcal{P}$

Asymptotic Notation – Big-O

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Definition

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$$f(n) \leq cg(n)$$

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- Note that $f(n) = O(n^4)$

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Time Complexity Classes

Let $t : \mathbb{N} \to \mathbb{N}$. Define time complexity class $TIME(t(n))$ as

$$TIME(t(n)) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time TM}\}$$

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Important

Time complexity depends on the exact model of computation

Dependence on Model of Computation

Theorem

For any function $t(n) \geq n$, every multi-tape TM (with $O(1)$ tapes) running in time $t(n)$ has an equivalent 1-tape TM running in time $O(t^2(n))$.

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- Convenient closure properties:

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- Polynomials grow much slower than exponentials:
  - $f(n) = n^3$: If $n = 1000$, $f(n) = 1,000,000,000$ – large, but not unreasonable for today's PCs
  - $f(n) = 2^n$: If $n = 1000$, $f(n) >$ number of atoms in the universe
- All "reasonable" deterministic computation models are polynomially equivalent
- Convenient closure properties:
  - $\text{poly}(n) + \text{poly}(n) = \text{poly}(n)$

Polynomial Time

Efficient Computation

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  - $\text{poly}(n) \cdot \text{poly}(n) = \text{poly}(n)$ (up to $O(1)$ multiplications)

Outline

Complexity Class $\mathcal{P}$

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$\mathcal{P}$ is the class of languages decidable in polynomial time on a 1-tape deterministic TM.

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Problems in $\mathcal{P}$

PATH problem

$PATH = \{\langle G, s, t \rangle \mid G \text{ is a directed graph that has a path from } s \text{ to } t\}$

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RELPRIME problem

$$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime integers}\}$$

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- We often more naturally think of computation as search problems (i.e., find a path from s to t)
- For some complexity classes, but not all, the two are equivalent – we will talk about this more later

Next Class

- Nondeterministic computation and the class $\mathcal{NP}$