# Foundations of Computing
## Lecture 21

Arkady Yerukhimovich

April 9, 2024

# Outline

1. Lecture 20 Review
2. A Review of $\mathcal{P}$ and $\mathcal{NP}$
3. Polynomial-Time Reductions
4. $\mathcal{NP}$-Completeness
5. $\mathcal{NP}$-Completeness Using Reductions

# Lecture 20 Review

- Verifying vs. Deciding
- The Complexity Class $\mathcal{NP}$

$$\mathcal{NP} = \bigcup_k NTIME(n^k)$$

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Both $\mathcal{P}$ and $\mathcal{NP}$ contain many useful languages

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## $\mathcal{NP}$-Completeness

There are problems in $\mathcal{NP}$ that are as hard as any other problem in $\mathcal{NP}$

# Outline

# Mapping Reductions

## Mapping Reduction

Language $A$ is mapping reducible to language $B$ ($A \leq_m B$) if there is a computable function $f : \Sigma^* \to \Sigma^*$, where for every $x$,

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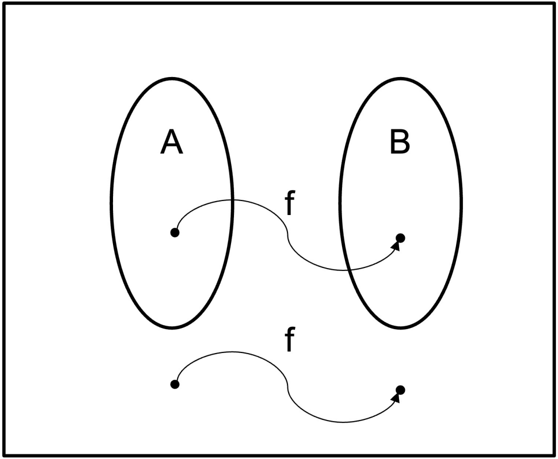
- Poly-time reductions give an efficient way to convert membership testing in $A$ to membership testing in $B$
- If $B$ has a poly-time solution so does $A$

# Poly-time Mapping Reductions




$f$ runs in time $poly(|x|)$ on all inputs $x$

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  $M'$ = On input $x$: Q: Is $x \in A$?
  1. Compute $f(x)$
  2. Run $M(f(x))$ and output whatever $M$ outputs

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  - If $x \in A$, $f(x) \in B$ so $M$ accepts
  - If $x \notin A$, $f(x) \notin B$, so $M$ rejects
  - Since both $f$ and $M$ are poly-time, $M(f(x))$ is also poly-time

# Using Poly-Time Reductions to Prove Hardness

**Theorem**

If $A \leq_P B$ and $A \notin \mathcal{P}$, then $B \notin \mathcal{P}$

1. Assume $B \in P \rightarrow \exists M$ that decides $B$

$M'(x)$

$M(f(x))$

# Outline

# $\mathcal{NP}$-Completeness

## Definition

A language $B$ is $\mathcal{NP}$-complete if

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- For every language $A \in \mathcal{NP}$, $A \leq_P B$

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If $B$ is $\mathcal{NP}$-complete and $B \leq_P C$ for $C \in \mathcal{NP}$, then $C$ is $\mathcal{NP}$-complete

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$$SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$$

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     - That is, $\phi$ corresponds to the Boolean logic done by this machine
     - Since any computation can be represented as a Boolean computation, this is always possible

# Execution of Turing Machine $M$ deciding $A$

start → poly(|x|)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| # | $q_0$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ | $\sqcup$ | $\cdots$ | $\sqcup$ | # |
| # | | | | | | | | | # |
| # | | | | | | | | | # |
| | | | | | | | | | |
| # | | | | | | | | | # |

Table: Tableau of configurations of $M$

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- $M$ accepts $x$ if a row of this tableau is in $q_{accept}$

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- The following equation $\phi_{i,j}^{cell}$ guarantees that a cell has a valid value

cell[i,j] ≠ s or cell[i,j] ≠ t

$$\phi_{i,j}^{cell} = \underbrace{\left(\bigvee_{s \in C} x_{i,j,s}\right)}_{\text{cell } i,j \text{ has at least 1 value}} \wedge \underbrace{\left(\bigwedge_{s,t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}})\right)}_{\text{cell } i,j \text{ has at most 1 value}}$$

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- For $1 \leq i, j \leq n^k$, and $s \in C$, let $x_{i,j,s} = 1$ if $cell[i,j] = s$
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$$\phi_{i,j}^{cell} = \underbrace{\left( \bigvee_{s \in C} x_{i,j,s} \right)}_{\text{cell } i,j \text{ has at least 1 value}} \wedge \underbrace{\left( \bigwedge_{s,t \in C, s \neq t} (\overline{x_{i,j,s}} \vee \overline{x_{i,j,t}}) \right)}_{\text{cell } i,j \text{ has at most 1 value}}$$

- Now, we just take the AND over all $n^{2k}$ cells in the tableau

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$$\phi_{start} = x_{1,1,\#} \wedge x_{1,2,q_0} \cdots \wedge x_{1,n^k,\#}$$

$x_{i,j,c} = 1$ if $cell[i,j] = c$

$\#\ q_0\ x_1\ x_2\ x_n\ \sqcup\ \sqcup\ \#$

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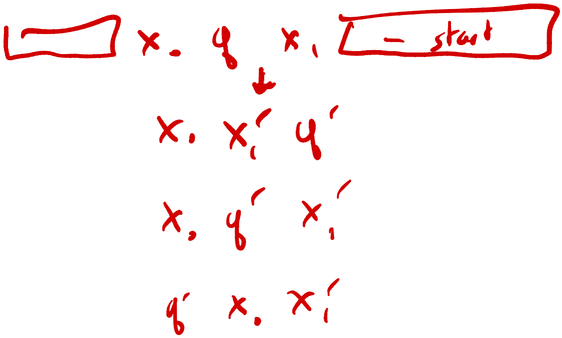

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   - Now just take the $\wedge$ over all possible 6-cell windows

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- Since $k = O(1)$, this is polynomial in $n$

# Outline

1. Lecture 20 Review
2. A Review of $\mathcal{P}$ and $\mathcal{NP}$
3. Polynomial-Time Reductions
4. $\mathcal{NP}$-Completeness
5. **$\mathcal{NP}$-Completeness Using Reductions**

$SAT \le_p A$

$A \in NP$

$\Rightarrow$

A is NP-complete

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## 3-SAT

$$\text{3SAT} = \{\langle \phi \rangle \mid \phi \text{ is a satisfiable 3-CNF formula}\}$$

Can show that 3SAT is $\mathcal{NP}$-complete using similar proof to SAT

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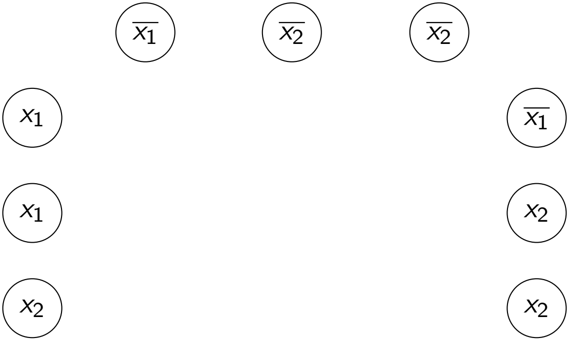

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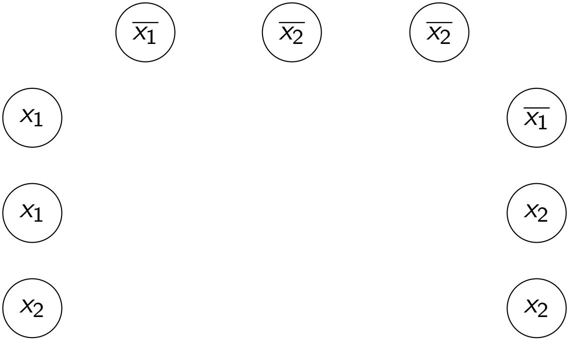


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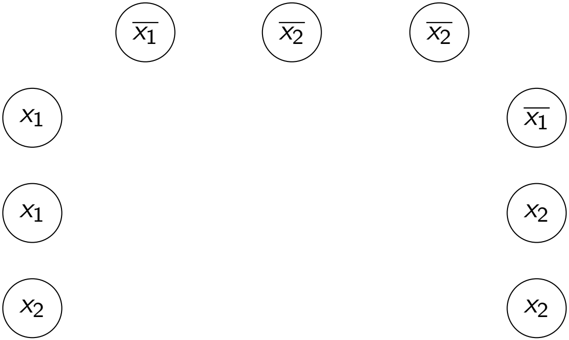


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