Foundations of Computing

Lecture 22

Arkady Yerukhimovich

April 11, 2024

Outline

Lecture 21 Review

- $\mathcal{P}$ and $\mathcal{NP}$
- Polynomial-Time Reductions
- $\mathcal{NP}$-completeness of SAT

Outline

What We Already Know

1. SAT is $\mathcal{NP}$-complete
2. 3-SAT is $\mathcal{NP}$-complete

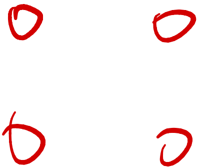
$$(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3)$$

3SAT $\leq_P$ CLIQUE

- Need to show reduction f from 3SAT formula ϕ to $\langle G, k \rangle$ where

1. Clique $\in$ NP

$w =$



2. 3SAT $\leq_p$ Clique

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$x_1 = 0$ $x_2 = 1$

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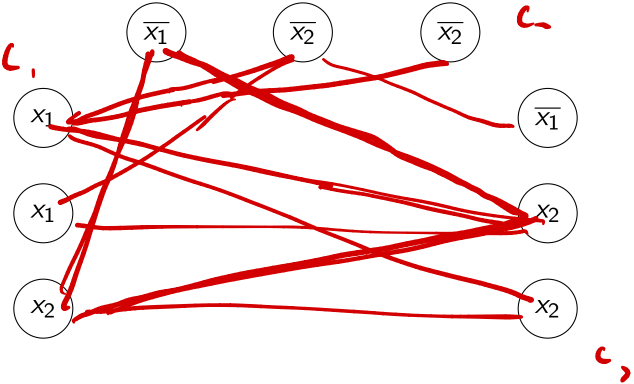

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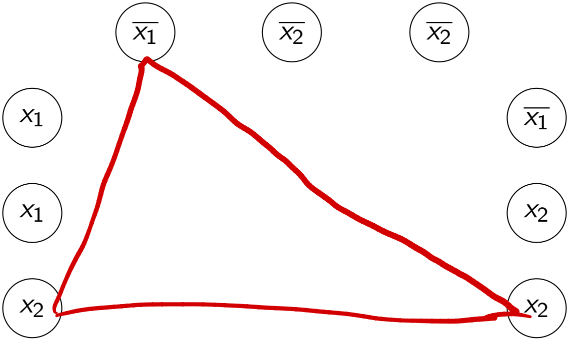

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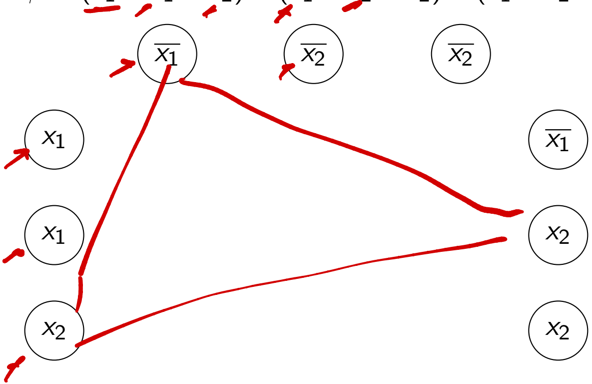

- If ϕ is satisfiable then G has a k-clique
- If G has a k-clique then ϕ is satisfiable

A Key Tool to Build Reductions

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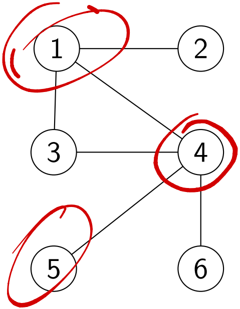
Vertex Covers

Given a graph $G = (V, E)$, a vertex cover is a subset of the nodes $C \subseteq V$ s.t. each edge in E has an end-point in V. C



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2. Show that 3-SAT $\leq_p$ VC

3-SAT $\leq_p$ VC

Goal: Show reduction f from 3-SAT to VC s.t.

- if ϕ is satisfiable, $f(\phi) = \langle G, k \rangle$ s.t. G has VC of size $\leq k$
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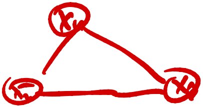
Variable gadget: For every variable x_1, draw pair of nodes

Clause gadget: For every (3-term) clause draw a triangle



Observations:

- For each variable need 1 node in cover
- For each triangle need at least 2 nodes
- Need to connect variables to clauses

3-SAT $\leq_p$ VC Example

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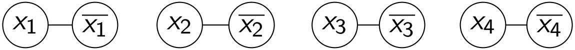

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2. No satisfying assignment implies smallest cover needs $|C| > 2c + v$

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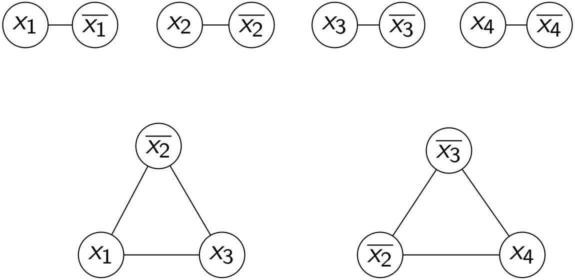

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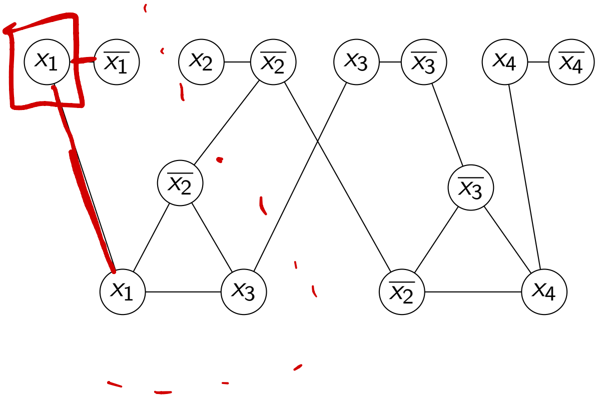

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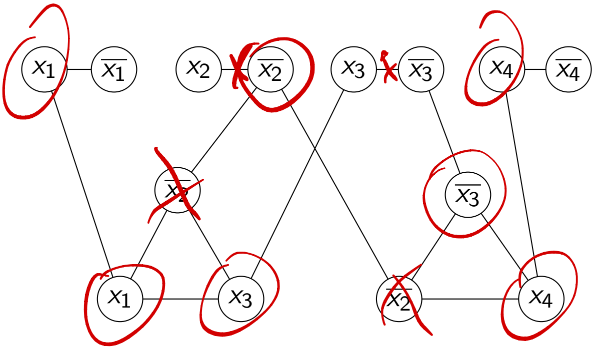

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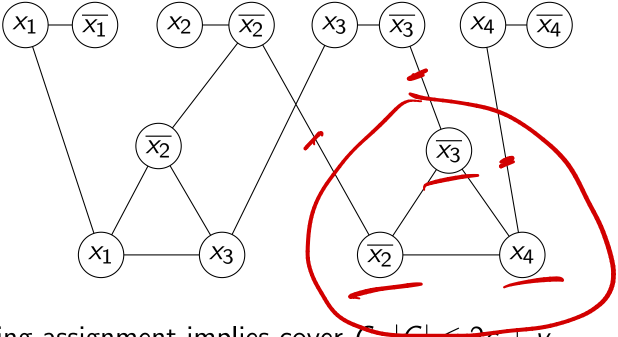

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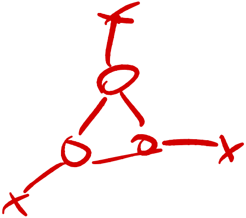

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  - We replaced each clause with a triangle and
  - each variable with a pair of nodes connected by an edge

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Definition

An undirected graph G is 3-colorable, if can assign colors $\{0, 1, 2\}$ to all nodes, such that no edges have the same color on both ends.

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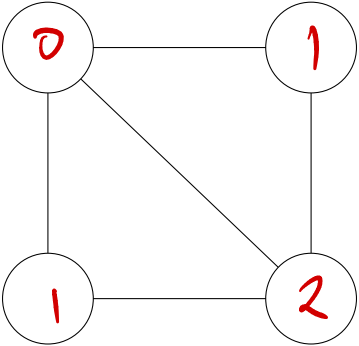

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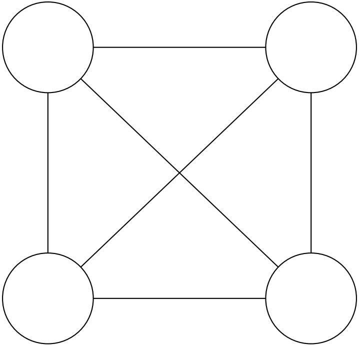

Goal: Prove than 3-Coloring is $\mathcal{NP}$-Complete

NAE-3SAT

NAE-kSAT Problem

NAE-kSAT $= \{\langle\phi\rangle \mid \phi$ is in k-CNF and ϕ has a satisfying assignment s.t. each clause has at least one 0 and at least one 1$\}$

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Lemma: If x is NAE-assignment of ϕ then $\overline{x}$ is NAE-assignment of ϕ

Proof:

- x must assign at least one 1 and at least one 0 to every clause
- $\overline{x}$ must also have at least one 1 and one 0 in every clause
- This means every clause is satisfied, and ϕ is satisfied since it's CNF

Goal

Prove that NAE-3SAT is $\mathcal{NP}$-complete: 3SAT $\leq_P$ NAE-3SAT

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    - If $S = 1$, then $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3} \vee 0)$ is also NAE-assignment. So, $(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) = 1$

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  - $(\Leftarrow)$ If $(x_1 \vee x_2 \vee z_i)$ and $(x_3 \vee x_4 \vee \overline{z_i})$ are both NAE

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Theorem

3SAT $\leq_P$ NAE-4SAT $\leq_P$ NAE-3SAT

NAE-3SAT $\leq_p$ 3-Coloring

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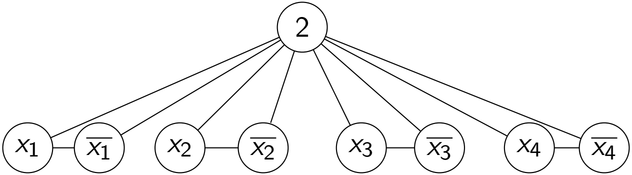

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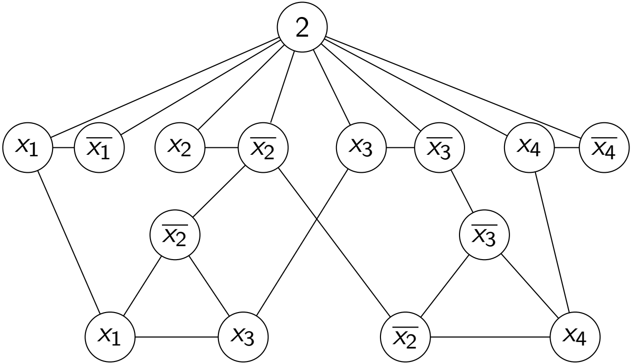

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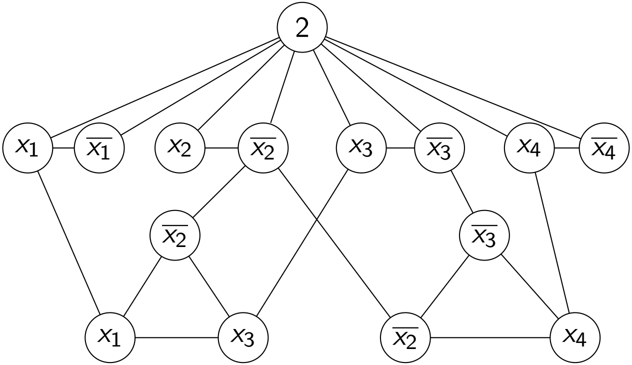

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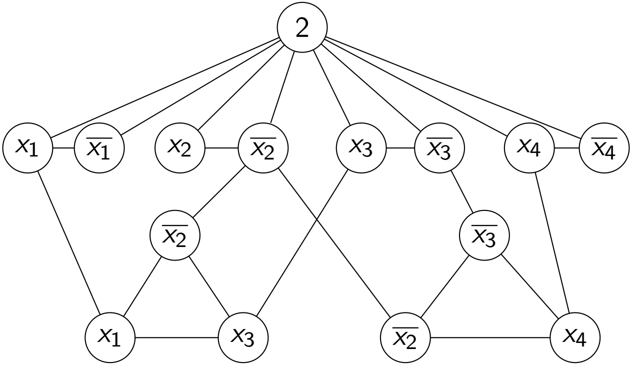

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3-Coloring is $\mathcal{NP}$-complete, but 2-Coloring $\in \mathcal{P}$

Outline

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Problems like UNSAT are in co-$\mathcal{NP}$

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$L \in \Sigma_2^p$ if there exists poly-time DTM V s.t. for $x \in L$, there exists a w_1 s.t. for all w_2, $V(x, w_1, w_2) = 1$

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We believe that there are infinitely many levels of the polynomial hierarchy and that $\Pi_i^p \neq \Sigma_i^p$ for $i > 0$, but can't prove it.

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Complexity Zoo

The complexity zoo (`https://complexityzoo.net/Complexity_Zoo`) now has 546 complexity classes.