# Foundations of Computing

## Lecture 24

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# Outline

# Lecture 23 Review

- 3-Coloring is $\mathcal{NP}$-complete
- Ladner's Theorem
- The class co-$\mathcal{NP}$

# $\mathcal{NP}$ vs co-$\mathcal{NP}$

### $\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM $V$ s.t. for $x \in L$ there exists a witness $w$ s.t. $V(x, w) = 1$

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Is $SAT \in$ co-$\mathcal{NP}$?

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- Can be an interactive procedure
- The verifier (and prover) can use randomness to decide whether to accept

# An Example – Aladdin's Cave

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- Can give proofs for languages not in $\mathcal{NP}$
- Interactive proofs can be much more efficient (e.g., shorter) than non-interactive ones
- Can have additional properties that traditional proofs cannot satisfy.
  - Zero-knowledge

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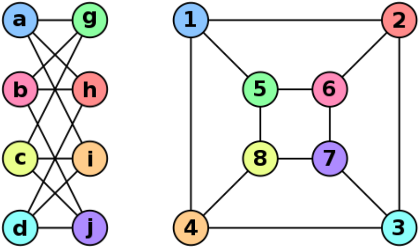

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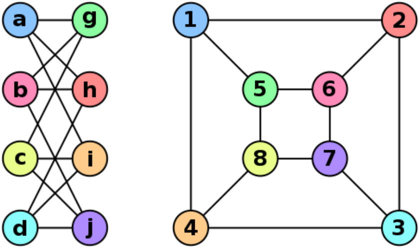


**Claim**

Graph Isomorphism $\in \mathcal{IP}$

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   - Thus, $\Pr[b' = b] = 1/2$

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- The power of interaction and randomness has allowed us to do what we couldn't do before

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3. $P^*$ wins with probability $\leq 1/2$ in each run, so

$$\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2^n$$

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- Suppose that $V$ is deterministic.
- What if you allow $V$ to be randomized?

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# Next Week

We have seen the power of interactive proofs in convincing a verifier of the truth of some statement.

**Question:**

What does the verifier learn from seeing the proof?