Foundations of Computing

Lecture 24

Arkady Yerukhimovich

April 17, 2025

Outline

Lecture 23 Review

- Vertex Cover is $\mathcal{NP}$-complete
- Ladner's Theorem
- The class co-$\mathcal{NP}$

$\mathcal{NP}$ vs co-$\mathcal{NP}$

$\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ there exists a witness w s.t. $V(x, w) = 1$

$\mathcal{NP}$ vs co-$\mathcal{NP}$

$\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ there exists a witness w s.t. $V(x, w) = 1$

co-$\mathcal{NP}$ – No instances are efficiently verifiable

$L \in$ co-$\mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \notin L$ there exists a witness w s.t. $V(x, w) = 1$

$\mathcal{NP}$ vs co-$\mathcal{NP}$

$\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ there exists a witness w s.t. $V(x, w) = 1$

co-$\mathcal{NP}$ – No instances are efficiently verifiable

$L \in$ co-$\mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \notin L$ there exists a witness w s.t. $V(x, w) = 1$

Comments:

- $L \in$ co-$\mathcal{NP} \iff \overline{L} \in \mathcal{NP}$
- co-$\mathcal{NP}$ contains the languages whose complement languages are in $\mathcal{NP}$

$\mathcal{NP}$ vs co-$\mathcal{NP}$

$\mathcal{NP}$ – Yes instances are efficiently verifiable

$L \in \mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \in L$ there exists a witness w s.t. $V(x, w) = 1$

co-$\mathcal{NP}$ – No instances are efficiently verifiable

$L \in$ co-$\mathcal{NP}$ if there exists poly-time DTM V s.t. for $x \notin L$ there exists a witness w s.t. $V(x, w) = 1$

Comments:

- $L \in$ co-$\mathcal{NP} \iff \overline{L} \in \mathcal{NP}$
- co-$\mathcal{NP}$ contains the languages whose complement languages are in $\mathcal{NP}$

Is $SAT \in$ co-$\mathcal{NP}$?

Outline

What is a Proof?

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula
- The Pythagorean Theorem is true

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula
- The Pythagorean Theorem is true
- ...

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula
- The Pythagorean Theorem is true
- ...

New Definition

A proof is any process at the end of which one party (the prover) can convince the other party (the verifier) of the truth of some statement x

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula
- The Pythagorean Theorem is true
- ...

New Definition

A proof is any process at the end of which one party (the prover) can convince the other party (the verifier) of the truth of some statement x

- A proof doesn't have to be a string

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

- x is a satisfiable formula
- The Pythagorean Theorem is true
- ...

New Definition

A proof is any process at the end of which one party (the prover) can convince the other party (the verifier) of the truth of some statement x

- A proof doesn't have to be a string
- Can be an interactive procedure

What is a Proof?

Traditional Definition

A proof is a string that convinces us of the truth of some mathematical statement x

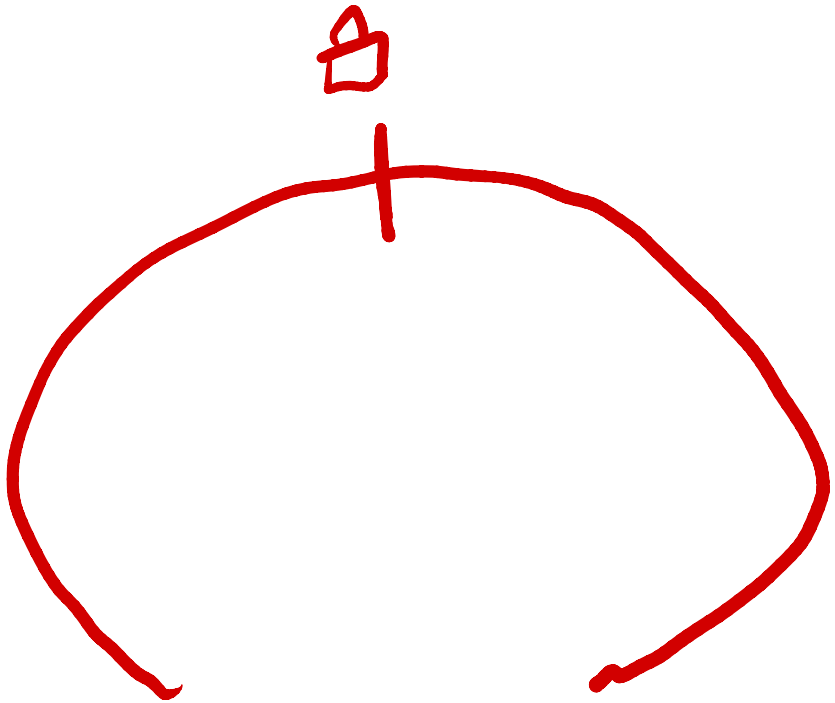
- x is a satisfiable formula
- The Pythagorean Theorem is true
- ...

New Definition

A proof is any process at the end of which one party (the prover) can convince the other party (the verifier) of the truth of some statement x

- A proof doesn't have to be a string
- Can be an interactive procedure
- The verifier (and prover) can use randomness to decide whether to accept

An Example – Aladdin's Cave



Why Interactive Proofs?

Question

We already know that all $L \in \mathcal{NP}$ have non-interactive proofs. Why study interactive ones?

Why Interactive Proofs?

Question

We already know that all $L \in \mathcal{NP}$ have non-interactive proofs. Why study interactive ones?

- Can give proofs for languages not in $\mathcal{NP}$

Why Interactive Proofs?

Question

We already know that all $L \in \mathcal{NP}$ have non-interactive proofs. Why study interactive ones?

- Can give proofs for languages not in $\mathcal{NP}$
- Interactive proofs can be much more efficient (e.g., shorter) than non-interactive ones

Why Interactive Proofs?

Question

We already know that all $L \in \mathcal{NP}$ have non-interactive proofs. Why study interactive ones?

- Can give proofs for languages not in $\mathcal{NP}$
- Interactive proofs can be much more efficient (e.g., shorter) than non-interactive ones
- Can have additional properties that traditional proofs cannot satisfy.
  - Zero-knowledge

Outline

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V \rangle(x) = 1] \leq 1/2$

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V \rangle(x) = 1] \leq 1/2$

Examples:

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V \rangle(x) = 1] \leq 1/2$

Examples:

- Aladdin's cave example from earlier

The Class $\mathcal{IP}$

Definition

$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V \rangle(x) = 1] \leq 1/2$

Examples:

- Aladdin's cave example from earlier
- $\mathcal{P} \subseteq \mathcal{IP}$

The Class $\mathcal{IP}$

Definition

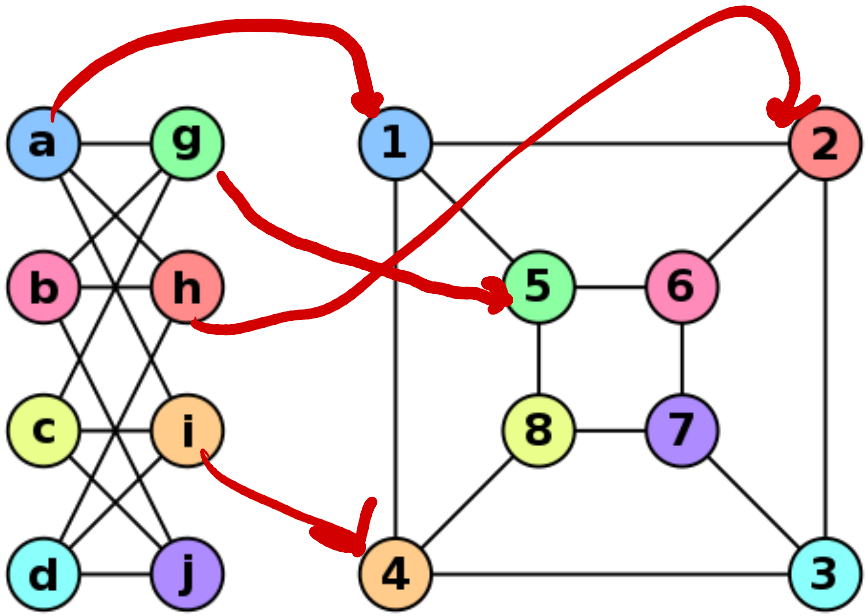
$L \in \mathcal{IP}$ if there exist a pair of interactive algorithms (P, V) with V being poly-time (in $|x|$) s.t.

1. (Completeness) If $x \in L$, then $\Pr[\langle P, V \rangle(x) = 1] = 1$
2. (Soundness) If $x \notin L$, then for any (possibly unbounded) P^*, we have $\Pr[\langle P^*, V \rangle(x) = 1] \leq 1/2$

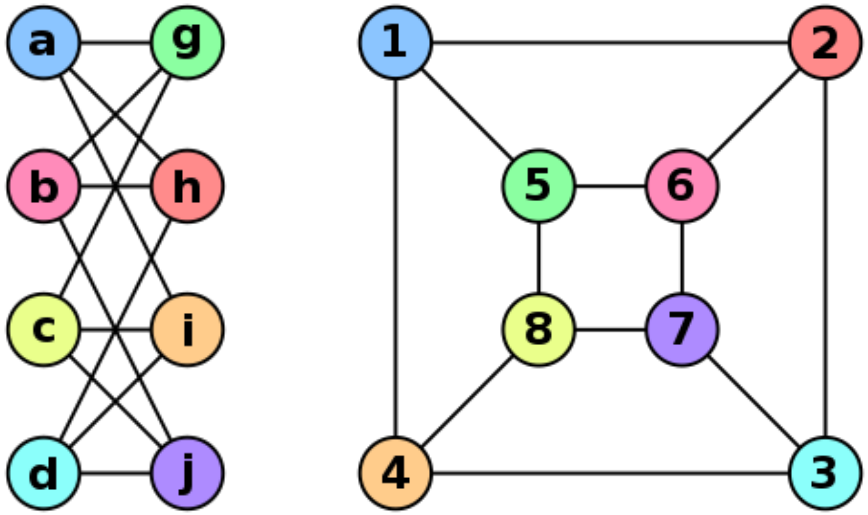
Examples:

- Aladdin's cave example from earlier
- $\mathcal{P} \subseteq \mathcal{IP}$
- $\mathcal{NP} \subseteq \mathcal{IP}$

Another Example – Graph Isomorphism



Another Example – Graph Isomorphism



Claim

Graph Isomorphism $\in \mathcal{IP}$

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0, 1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$
2. (Soundness) Suppose that G_0 and G_1 are isomorphic

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$
2. (Soundness) Suppose that G_0 and G_1 are isomorphic
   - Then G^* is isomorphic to both G_0 and G_1

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$
2. (Soundness) Suppose that G_0 and G_1 are isomorphic
   - Then G^* is isomorphic to both G_0 and G_1
   - P has no way to tell which one V started from

Graph Non-Isomorphism

Question

How can we prove that two graphs G_0 and G_1 are NOT isomorphic?

The Protocol:

1. V chooses $b \leftarrow \{0,1\}$, and applies a random permutation π to the vertices of G_b and sends this graph G^* to P
2. P determines if G^* is isomorphic to G_0 and sends $b' = 0$ if so, or $b' = 1$ otherwise back to V
3. V accepts if $b' = b$

Why This Works:

1. (Completeness) Suppose that G_0 and G_1 are not isomorphic.
   - Then G^* can only be isomorphic to one of the two graphs
   - P can perfectly determine which one this is
   - So $\Pr[b' = b] = 1$
2. (Soundness) Suppose that G_0 and G_1 are isomorphic
   - Then G^* is isomorphic to both G_0 and G_1
   - P has no way to tell which one V started from
   - Thus, $\Pr[b' = b] = 1/2$

Important Takeaways

Important Takeaways

- GNI $\in$ co-$\mathcal{NP}$

Important Takeaways

- GNI $\in$ co-$\mathcal{NP}$
- It is not believed that there is a short witness w s.t. $V((G_0, G_1), w) = 1$ if G_0 and G_1 are not isomorphic.
  I.e., GNI $\notin \mathcal{NP}$

Important Takeaways

- GNI $\in$ co-$\mathcal{NP}$
- It is not believed that there is a short witness w s.t. $V((G_0, G_1), w) = 1$ if G_0 and G_1 are not isomorphic.
  I.e., GNI $\notin \mathcal{NP}$
- The power of interaction and randomness has allowed us to do what we couldn't do before

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v \rangle(x) = 1] \leq 1/2$$

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v \rangle(x) = 1] \leq 1/2$$

What if we don't want malicious prover to win so often?

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v \rangle(x) = 1] \leq 1/2$$

What if we don't want malicious prover to win so often?

Soundness Amplification

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v \rangle(x) = 1] \leq 1/2$$

What if we don't want malicious prover to win so often?

Soundness Amplification

1. Run the proof n times sequentially on same input x, but different randomness

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v \rangle(x) = 1] \leq 1/2$$

What if we don't want malicious prover to win so often?

Soundness Amplification

1. Run the proof n times sequentially on same input x, but different randomness
2. Accept if ALL proofs accept

Boosting Soundness

So far, we defined soundness as:

$$\Pr[\langle P^*, v\rangle(x) = 1] \leq 1/2$$

What if we don't want malicious prover to win so often?

Soundness Amplification

1. Run the proof n times sequentially on same input x, but different randomness
2. Accept if ALL proofs accept
3. P^* wins with probability $\leq 1/2$ in each run, so

$$\Pr[\langle P^*, V\rangle(x) = 1] \leq 1/2^n$$

Outline

Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$f(x) = x^3 - 6x^2 + 11x - 7 = (x-1)(x-2)(x-3)$$

Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$f(x) = x^3 - 6x^2 + 11x - 7 = (x-1)(x-2)(x-3)$$

Properties:

- A root of a polynomial f is a value x s.t. $f(x) = 0$

Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$f(x) = x^3 - 6x^2 + 11x - 7 = (x-1)(x-2)(x-3)$$

Properties:

- A root of a polynomial f is a value x s.t. $f(x) = 0$
- The degree of a polynomial $f(x)$ is the maximum exponent in f

Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$f(x) = x^3 - 6x^2 + 11x - 7 = (x-1)(x-2)(x-3)$$

Properties:

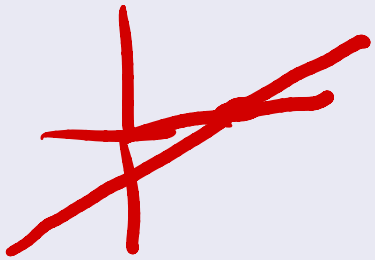
- A root of a polynomial f is a value x s.t. $f(x) = 0$
- The degree of a polynomial $f(x)$ is the maximum exponent in f
- A polynomial of degree d has at most d roots

Another Example – Polynomial Identity Testing

Polynomial

A polynomial is an equation in one-variable

$$f(x) = x^3 - 6x^2 + 11x - 7 = (x-1)(x-2)(x-3)$$



Properties:

- A root of a polynomial f is a value x s.t. $f(x) = 0$
- The degree of a polynomial $f(x)$ is the maximum exponent in f
- A polynomial of degree d has at most d roots
  - unless $f(x) = 0$

Another Example – Polynomial Identity Testing

PIT Problem

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

The rules:

- V is allowed to query $f(x)$ at points x of its choice

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

The rules:

- V is allowed to query $f(x)$ at points x of its choice
- P is required to answer honestly, but

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

The rules:

- V is allowed to query $f(x)$ at points x of its choice
- P is required to answer honestly, but
- P knows V's strategy (i.e., how he chooses the points x)

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

The rules:

- V is allowed to query $f(x)$ at points x of its choice
- P is required to answer honestly, but
- P knows V's strategy (i.e., how he chooses the points x)

Question: What should V do? How many queries does he need?

Another Example – Polynomial Identity Testing

PIT Problem

- Prover P chooses a degree d polynomial f and wants to prove that

$$\forall x, f(x) = 0$$

- Completeness: If $f(x) = 0$, V should accept after interacting with P
- Soundness: If $f(x) \neq 0$, V should reject

The rules:

- V is allowed to query $f(x)$ at points x of its choice
- P is required to answer honestly, but
- P knows V's strategy (i.e., how he chooses the points x)

$\frac{d}{101}$

Question: What should V do? How many queries does he need?

- Suppose that V is deterministic.
- What if you allow V to be randomized?

The Power of Randomness in Interactive Proofs

- By allowing V to be randomized we went from $d+1$ queries to 1 query

The Power of Randomness in Interactive Proofs

- By allowing V to be randomized we went from $d+1$ queries to 1 query
- We have strong evidence that derandomizing PIT will be very hard – it implies strong complexity results that we have no idea how to prove

The Power of Randomness in Interactive Proofs

- By allowing V to be randomized we went from $d+1$ queries to 1 query
- We have strong evidence that derandomizing PIT will be very hard – it implies strong complexity results that we have no idea how to prove

Take away

Randomness and interaction are key to the power of $\mathcal{IP}$

The Power of Randomness in Interactive Proofs

- By allowing V to be randomized we went from $d+1$ queries to 1 query
- We have strong evidence that derandomizing PIT will be very hard – it implies strong complexity results that we have no idea how to prove

Take away

Randomness and interaction are key to the power of $\mathcal{IP}$

Next Week

We have seen the power of interactive proofs in convincing a verifier of the truth of some statement.

Question:

What does the verifier learn from seeing the proof?