Foundations of Computing

Lecture 25

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April 23, 2024

Outline

Lecture 24 Review

- Interactive Proofs
- Proof for Graph Non-Isomorphism
- Polynomial Identity Testing

Outline

Reviewing the Definition of $\mathcal{IP}$

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A New Property

We say that a proof is *zero-knowledge* if the verifier learns nothing (other than the truth of the statement) from seeing the proof.

An Example – Where's Waldo

An Example

A Second Example – Puppy and Panda

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What does it mean for a machine to learn nothing from a proof?

Answer: Whatever it can (efficiently) compute after seeing the proof, it could have efficiently computed before seeing the proof.

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- Thus, the proof is zero-knowledge: V learns nothing more than that $x \in L$
- IMPORTANT: $VIEW_V^*$ and $S(x)$ are both distributions, not values. So, equality is of distributions

Outline

Where's Waldo

Puppy and Panda

flip

← flip

S

1. choose flip / no flip
output the

$VIEW_V = (win, win)$

Graph Isomorphism

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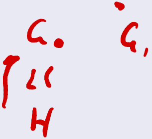

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2. Soundness: Suppose G_0 is not isomorphic to G_1, so there is no such π. Then, if $b \neq b'$, there is no permutation that P can give that V will accept

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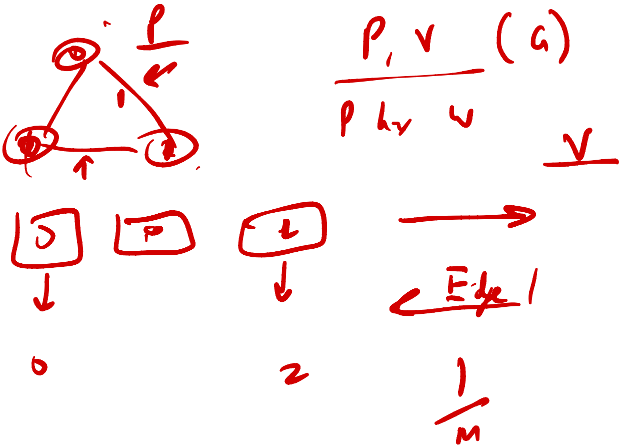
- When S stops, he produces a perfect simulation

Graph 3-Coloring



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ZK Proofs enable privacy-preserving transactions on a public Blockchain!