Foundations of Computing

Lecture 2

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Outline

1. Academic Integrity Policies
2. Lecture 1 Review
3. Language accepted by M
4. Quiz Solutions
5. Building DFAs
6. Proving Correctness of a DFA

Homework Policies

Important

Any work you submit MUST be your own!

You may do the following:

- discuss general concepts/questions with others
- discuss similar problems not in homework (e.g., from book)

You may NOT do the following:

- Copy or provide answers to any hw problems to others
- Use ChatGPT or any other LLM to produce your answers
- Search the web for solutions or use services like chegg.com or StackExchange

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Lecture 1 Review

- Syllabus review and course details
- Strings and languages
- Finite automata

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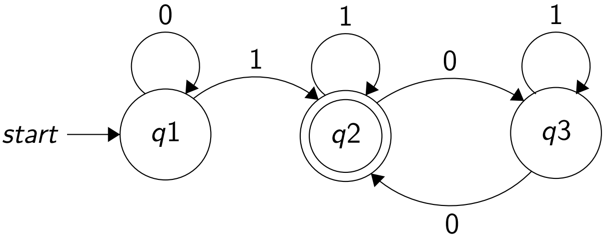
Language accepted by M



Accepting a string

- M accepts a string x (over Σ) if $M(x)$ stops in an accept state
- What strings does M accept?

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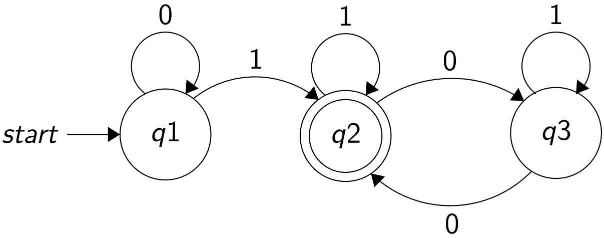

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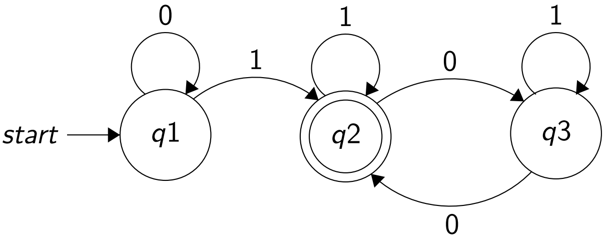
Accepting a language

- M accepts/decides a language L if it accepts:
  - ALL strings in L, and
  - NO strings not in L
- Every M accepts exactly one language $L(M)$

What language does M accept?



$L(M)$:

- String must contain at least one 1
- After the first string of 1's, there must be an even number of 0's or no 0's

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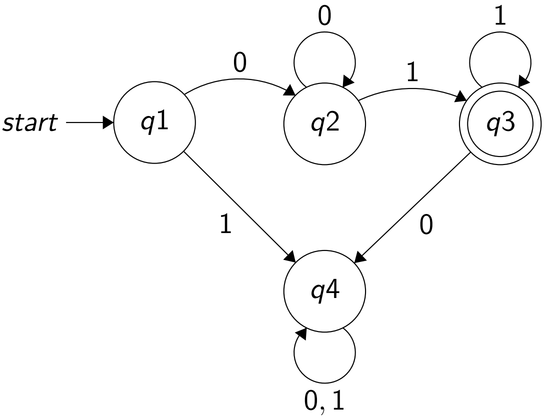
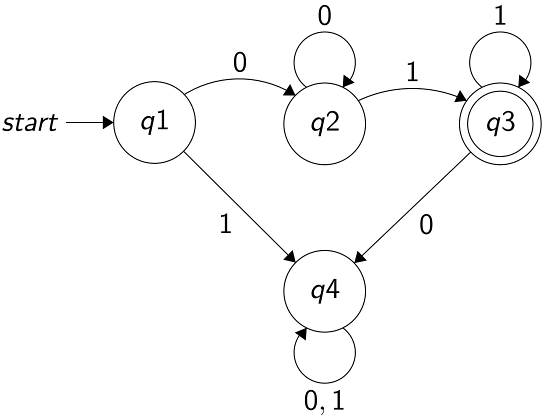
Quiz Solutions



- Does M accept 00011?: Yes

Quiz Solutions



- Does M accept 00011?: Yes
- Does M accept 01100? No

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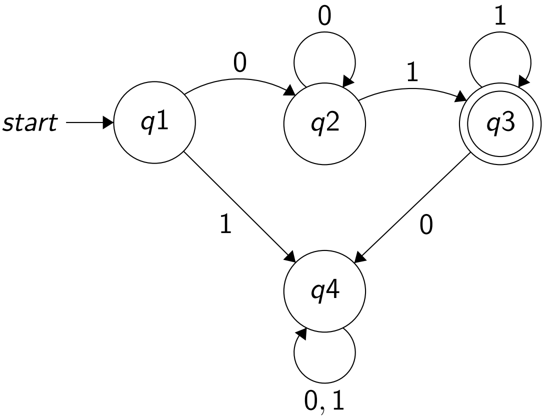

- Does M accept 00011?: Yes
- Does M accept 01100? No
- Describe the language $L(M)$: all strings with one or more 0s followed by one or more 1s

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Important Rules of Deterministic Finite Automata

Deterministic Finite Automata

- Transition function must be fully defined:
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Important: Deterministic means that the execution of M on any input must be fully specified.

DFA as an Algorithm

DFA Execution

1. Read next input symbol and use transition function to determine next step until run out of input symbols
2. If stop in accept state, then output 1

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- Each state stores a summary of the input seen so far
- Next state depends on the current state and the next symbol
- Think of this as an "if" statement

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Important

Since $|Q|$ is finite, input string may be longer than number of states

- Cannot just store the entire string

Example 1

Problem

Build a DFA that accepts

$$L = \{w | w \in \{0,1\}^* \text{ and } w \text{ contains the substring } 101\}$$

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- If see a 0:
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  - but can be second character if previous symbol was a 1
- If see a 1:
  - this can be the first character of 101
  - or, it can be the last character if we previously saw 10 – in this case, we should accept

Example 1 – The Algorithm

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Algorithm:

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3. Step 3:
   - If read a 0, goto step 1 – this is not 101, time to start over
   - If read a 1, goto step 4 – we have seen 101

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4. Step 4:
   - On any input, stay in step 4 and accept

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The DFA

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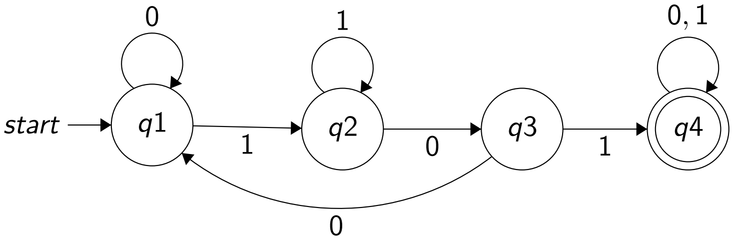

1. $q1$ – not yet read first 1 in 101
2. $q2$ – last input was a 1, could be start of 101
3. $q3$ – have read 10
4. $q4$ – have read 101

Trap States

A useful tool for designing DFAs:

- Trap states allow you to "reject" as soon as you know that $w \notin L$

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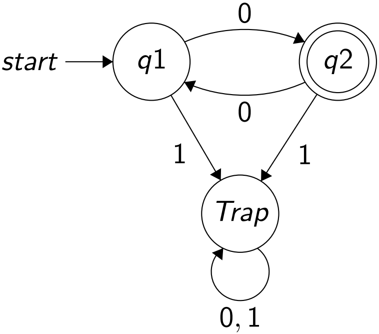

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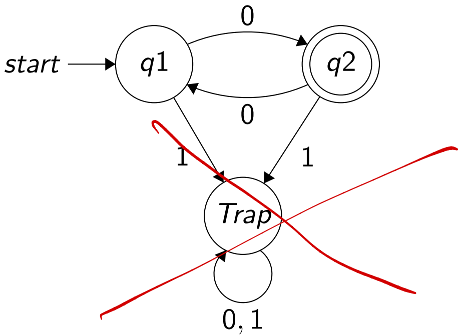

For convenience

You can omit edges from transition diagram that point to the trap state

Example 2

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Build a DFA that accepts:
$L = \{w | w \in \{0, 1\}^*$ and has an even number (≥ 2) 1's followed by an odd number (≥ 1) 0's$\}$

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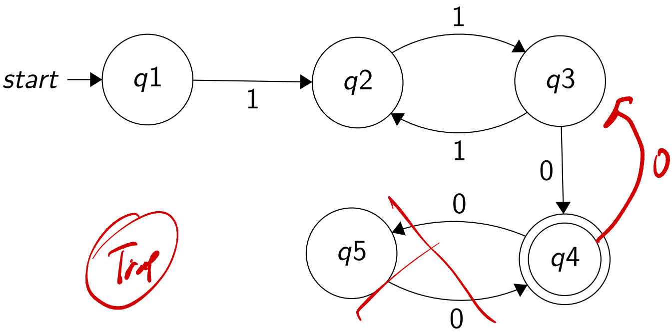

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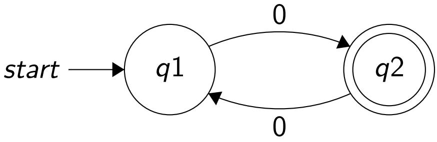
Another Example

Consider the following DFA M



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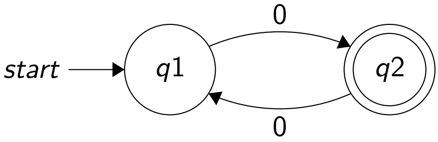

Theorem: This DFA recognizes

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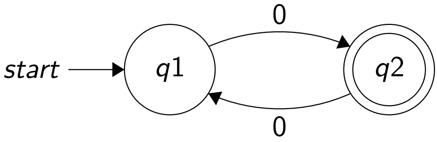

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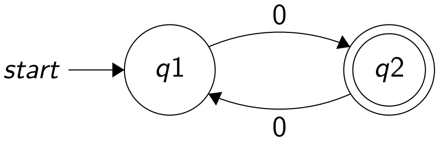
Proof:

- Need to prove that $L = L(M)$
- Instead we prove the $L \subseteq L(M)$ and $L(M) \subseteq L$

$L \subseteq L(M)$



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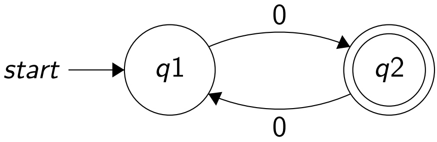

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Claim: Every $w \in L$ will cause M to accept (i.e., stop in $q2$).

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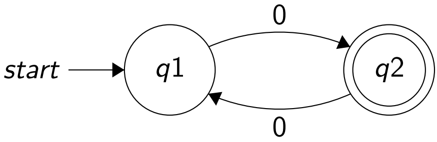

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If $|w| = 1$ and $w \in L$ then $w = 0$ and $M(w) = 1$

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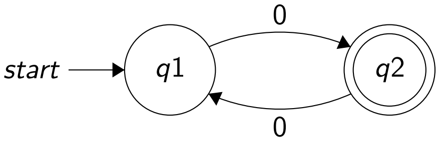

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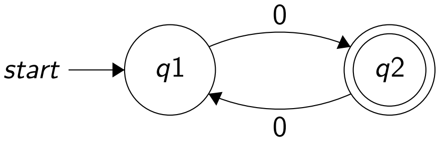

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Proof by Induction:
Consider $|w| = k + 2$ and let w' be the prefix of w of length k.
By hypothesis $\delta^*(q1, w') = q2$, and last two bits of w must be 0's
Hence $\delta^*(q1, w) = q2$

$L(M) \subseteq L$

Claim: Every w accepted by M is in L.

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Proof by contradiction:
Assume there exists a string w accepted by M that is not in L

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Proof:

1. w cannot have a 1, as any such input will not stop in $q2$
2. By similar proof to before, any w with even number of 0's must stop in $q1$
3. Contradiction!