Foundations of Computing

Lecture 4

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January 25, 2024

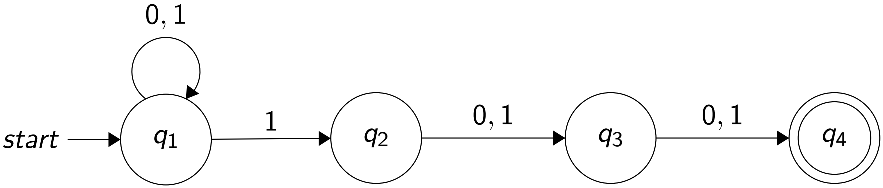
Outline

Lecture 3 Review

- Regular Languages
- Nondeterministic Finite Automata
- Understanding Nondeterminism

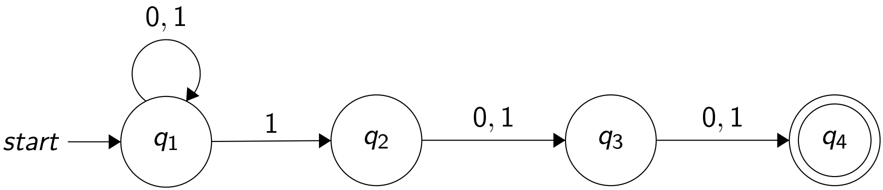
Outline

NFA Example 1



Question: What is $L(M)$?

NFA Example 1



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Answer: Strings in $\{0,1\}^*$ with a 1 as third from the end

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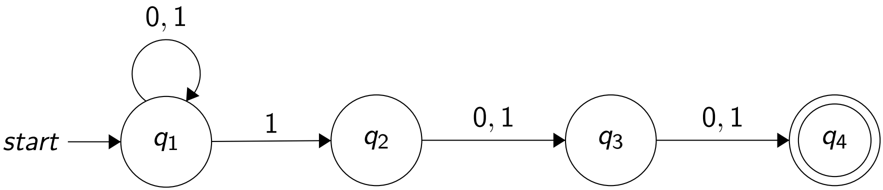

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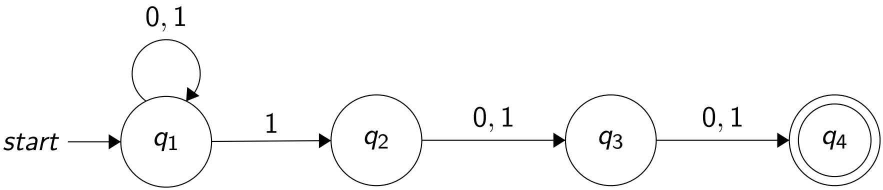

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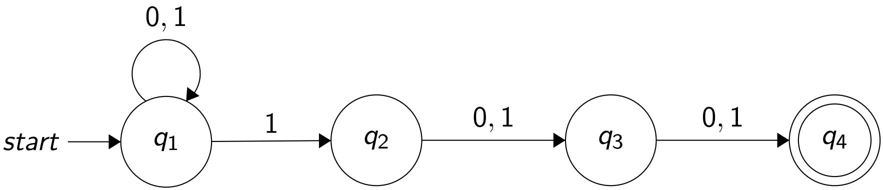

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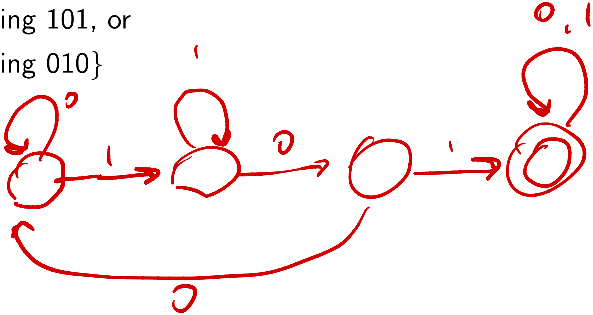
How does it work?

- M waits in q_1 until it "guesses" that it is 3 symbols from the end
- Uses the rest of the states to verify that 1 is third from the end
- DFA doing the same thing would have to track the last three bits seen – requires 8 states

Example 2 – OR statement

$L = \{x | x \in \{0,1\}^*$ and x contains

1. the substring 101, or
2. the substring 010$\}$



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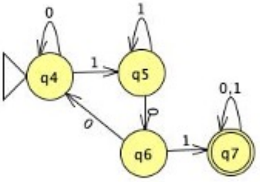

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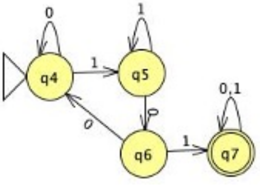
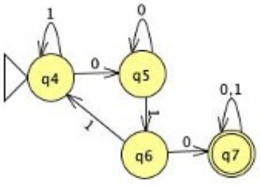


DFA for prop. (1)



DFA for prop. (2)

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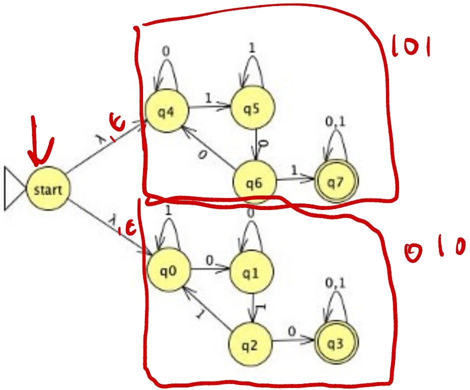

NFA for L

NFA Summary

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- Only need to verify that inputs have correct form
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Question

Are NFAs more powerful than DFAs?

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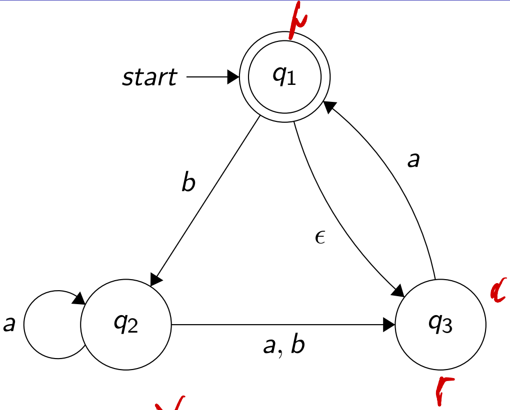

1. Does N accept $w = \epsilon$? Yes
2. Does N accept $w = aaa$? Yes
3. Does N accept $w = babba$? No
4. Does N accept $w = abaaba$? Yes

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DFAs == NFAs

Theorem

For every NFA N there exists an equivalent DFA M

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- Let node of DFA M represent set of “highlighted” nodes

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A little more detail:

- Let node of DFA M represent set of “highlighted” nodes
- Define δ to move to new set of highlighted nodes
- Accept states are ones in which at least one node is an accept node
- Can deal with ϵ edges by “placing more fingers” on resulting nodes

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Look at transitions from all states in set R and map to set that gives results of all these transitions

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   $$\delta'(R, a) = \cup_{r \in R} \delta(r, a)$$

   Look at transitions from all states in set R and map to set that gives results of all these transitions
3. $q_0' = \{q_0\}$
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$
   Accept if any state in R is an accept state

Handling ϵ transitions

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Formally:

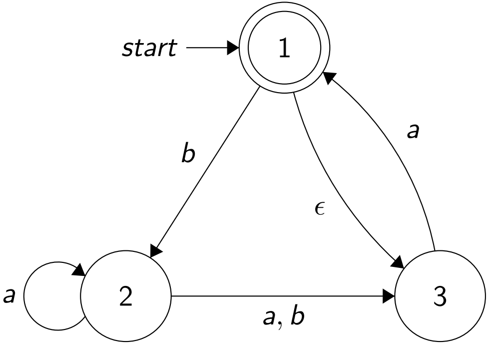
1. Let $E(R) = \{q | q \text{ can be reached from } R \text{ along } \epsilon \text{ arrows}\}$
2. Define extended transition function

$$\delta'(R, a) = \cup_{r \in R} E(\delta(r, a))$$

Map to set of states that can be reached on input a or $a\epsilon$

An Example: NFA $\rightarrow$ DFA

NFA $N = (Q, \Sigma, \delta, q_0, F)$



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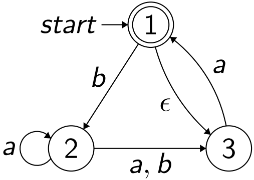

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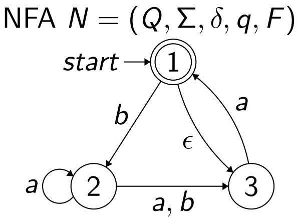

1. states: $Q' = P(Q) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
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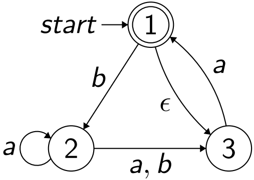

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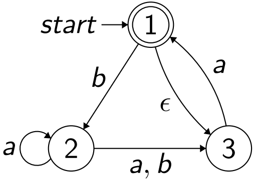

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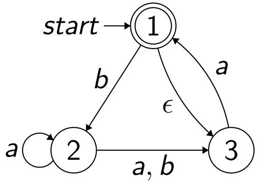

4. Transition function δ':

$\delta'(\emptyset, a) =$ ∅
$\delta'(\{1\}, a) =$ ∅
$\delta'(\{2\}, a) =$ {2, 3}
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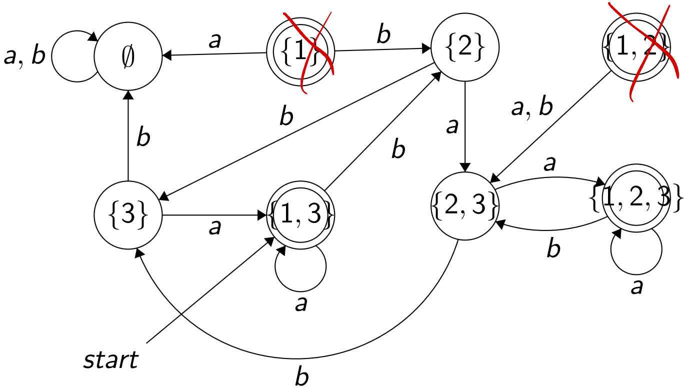

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A Useful Corollary

Recall that:

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We can now use NFAs to argue the properties of regular languages

Closure Under Union

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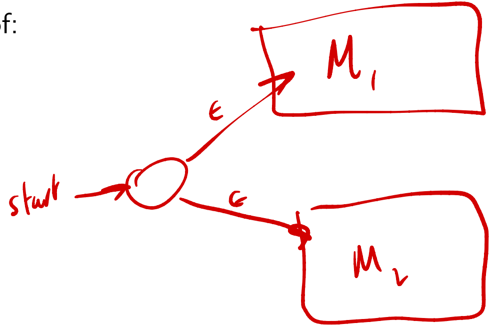
If L_1 and L_2 are both regular languages then $L_1 \cup L_2$ is also regular

$L_1 \cup L_2$ is the language consisting of all strings either in L_1 or L_2

Proof:

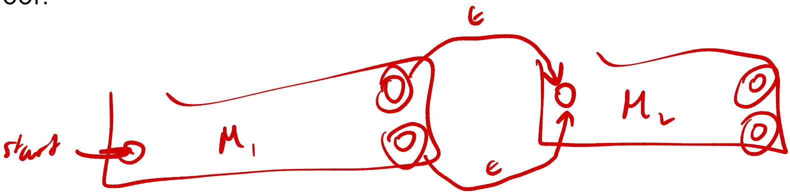


Closure Under Concatenation

If L_1 and L_2 are both regular languages then $L_1 \circ L_2$ is also regular

$L_1 \circ L_2 = \{xy | x \in L_1 \text{ and } y \in L_2\}$

Proof:



Closure Under the Star Operation

Closure Under Star Operation

If L is a regular languages then L^* is also regular

$L^* = \{0 \text{ or more strings from } L\}$

Proof: