Foundations of Computing

Lecture 5

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Outline

Lecture 4 Review

- More NFAs
- Equivalence of NFAs and DFAs
- NFAs for union, composition, and star – closure of regular languages

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Regular Expressions

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You've seen this before

Regular expressions very useful in compilers, and string search (e.g., grep)

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6. (R_1^*) – 0 or more repetitions of R_1 where R_1 is a regular expression

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- $1^*\emptyset = \emptyset$
- $\emptyset^* = \{\epsilon\}$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 = \{w | w \text{ starts and ends with the same symbol}\}$

Languages to Regular Expressions Examples

Consider languages over the alphabet $\{0, 1, 2\}$

1. $L_1 = \{w | w \text{ has 2 consecutive 0's}\}$

   $\Sigma^* 00 \Sigma^*$

2. $L_2 = \{w | w \text{ has a substring 101 and ends in 22}\}$

   $\Sigma^* 101 \Sigma^* 22$

3. $L_3 = \{w | w \in L_1 \text{ or } w \in L_2\}$

   $(R_1) \cup (R_2)$

   $(\Sigma^* 00 \Sigma^*) \cup (\Sigma^* 101 \Sigma^* 22)$

Question:

What does this have to do with FAs and regular languages?

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Regular Expressions == Regular Languages == NFA

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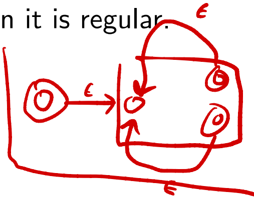

1. $R = a$ for some $a \in \Sigma$

2. $R = \epsilon$

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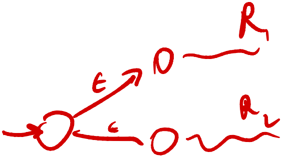

4. $R = R_1 \cup R_2$
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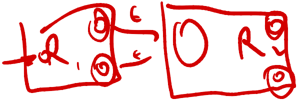
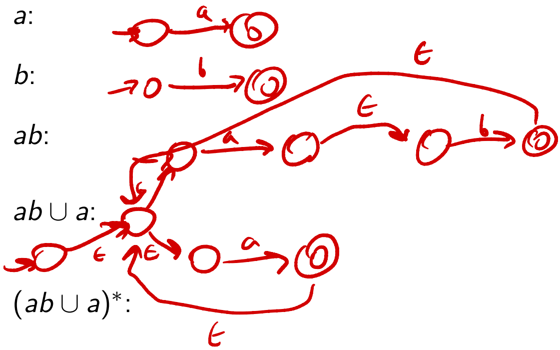


An Example

Problem: Convert $(ab \cup a)^*$ to an NFA

In English: Either "ab" or "a" repeated 0 or more times

- a:
- b:
- ab:
- $ab \cup a$:
- $(ab \cup a)^*$:



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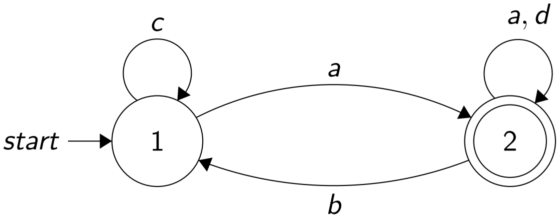

Question

How do we represent L by a regular expression?

Step 1: NFA $\rightarrow$ generalized NFA

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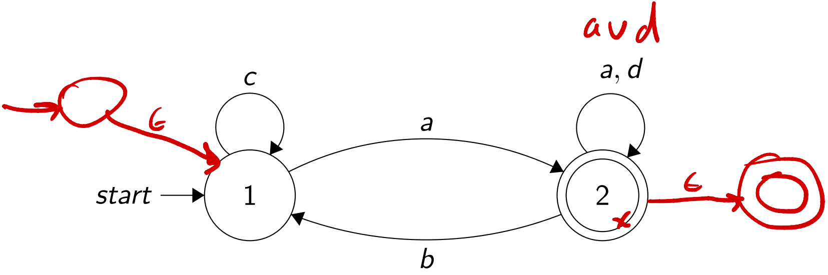
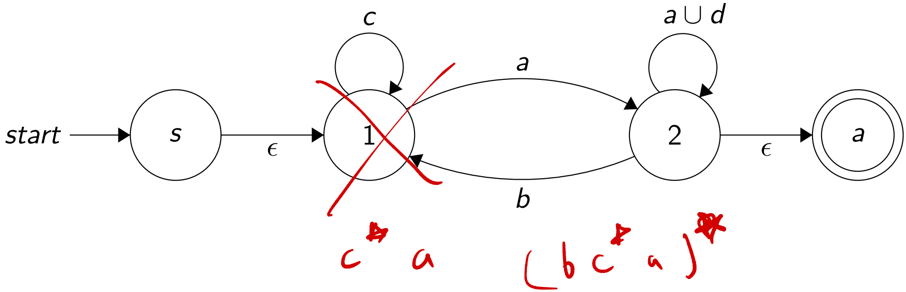


Step 2: Node Elimination – Remove Node 1

Remove nodes one-by-one (in any order) until only start and accept states left:

- Need to update reg. exp.'s for all paths through removed nodes



Step 2: Node Elimination – Remove Node 2

Remove nodes one-by-one (in any order) until only start and accept states left:

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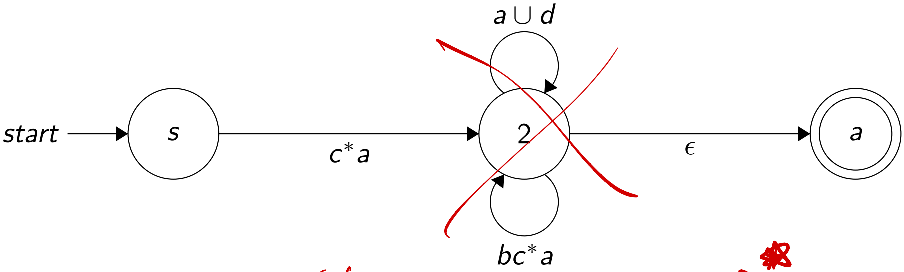
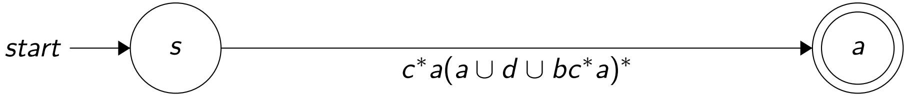


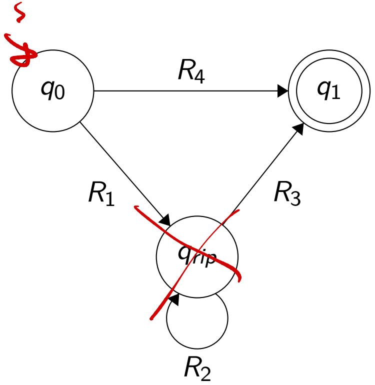
$(c^* a)((a \cup d) \cup (bc^* a))^*$

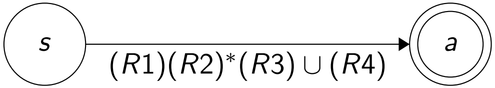
We are Done

Output label of final edge from start to accept state.



Generalized Node Elimination





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Base Case: For $|G| = 2$, G consists of start and accept states and arrow between them. The label on this arrow exactly describes the language of strings accepted by G.

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For any GNFA G, G'=NODE-ELIMINATE(G) is equivalent to G

Inductive step: Assume true for $|G| = k - 1$, prove true for $|G| = k$. (i.e., prove that $G' = G$)

- Assume some w s.t. $G(w) = 1$, then on input w, G goes through

$$q_{start}, q_1, q_2, \ldots, q_{accept}$$

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  - If the accepting path would not have gone through q_{rip}, then G must also have the same path to accept w

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- Build NFA M corresponding to each clause
- Since we already showed how to build NFA to show closure, can convert that to regular expression to prove the claim.