Foundations of Computing
Lecture 7

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Outline

Lecture 6 Review

- Nonregular languages
- Proving The NFA pumping lemma
- Using the pumping lemma

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- Using the pumping lemma

Today

- Some more examples proving languages are not regular
- Going beyond regular languages

HW2 Problem 4

Let L be a regular language, prove that the following languages are regular.

1. $NOPREFIX(L) = \{w \in L|$ no proper prefix of w is a member of $L\}$
2. $NOEXTEND(L) = \{w \in L| w$ is not a proper prefix of any string in $L\}$

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Example:

- $L = \{00, 11, 001, 101\}$
- $NOPREFIX(L) = \{00, 11, 101\}$
- $NOEXTEND(L) = \{11, 001, 101\}$

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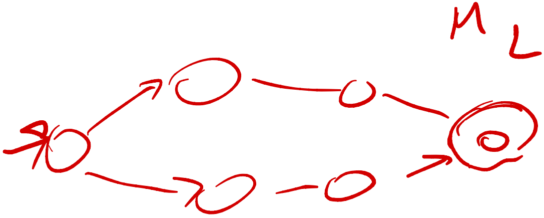

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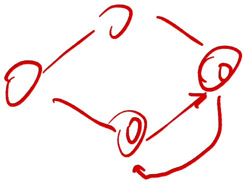

Outline

The Regular Language Pumping Lemma

Pumping Lemma

If L is a regular language, then there exists an integer p (the pumping length) where any string $w \in L$ such that $|w| \geq p$ can be divided into three pieces $w = xyz$ satisfying:

1. For each $i \geq 0$, $xy^iz \in L$
2. $|y| > 0$, and
3. $|xy| \leq p$

$\exists p \; \forall |w| \geq p \; \exists$ partition xyz s.t. $\forall i \; xy^iz \in L$

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The Proof Procedure

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3. Choose $w \in L$ with $|w| \geq p$

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2. Use pumping lemma to guarantee pumping length p, s.t. all w with $|w| > p$ can be pumped – Note: proof must work for all p
3. Choose $w \in L$ with $|w| \geq p$
4. Demonstrate that w cannot be pumped
   - For each possible division $w = xyz$ (with $|y| > 0$ and $|xy| \leq p$), find an integer i such that $xy^iz \notin L$

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4. Demonstrate that w cannot be pumped
    - For each possible division $w = xyz$ (with $|y| > 0$ and $|xy| \leq p$), find an integer i such that $xy^iz \notin L$
5. Contradiction!!!

Prior Examples

We've already seen how to prove:

- $L = \{0^n 1^n | n \geq 0\}$ is not regular
- $L = \{w | w \text{ has an equal number of 0s and 1s}\}$ is not regular

In both proofs, we picked $w = 0^p 1^p$
Easy to show that this string cannot be pumped

A More Challenging Example

Consider $L = \{0^m 1^n | m \neq n\}$, prove L is not regular

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   That is, $xy^i z = 0^{m'} 1^{n'}$ with $m' = n'$.

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Question

What w should we choose?

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Pick a w s.t. for all partitions $w = xyz$, for some $i \geq 0$, $xy^i z = 0^{m'} 1^{m'}$.

1. Suppose we choose $w = 0^p 1^{p+1}$, then since $|xy| \leq p$, $x = 0^\alpha$, $y = 0^\beta$, $z = 0^{p-(\alpha+\beta)} 1^{p+1}$

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$$xy^k z = 0^{\underbrace{\alpha}_{x}+\underbrace{k\beta}_{k\cdot\beta}+\underbrace{p-(\alpha+\beta)}_{z}} 1^{p+1}.$$

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$$xy^k z = 0^{\underline{\alpha+k\beta+p-(\alpha+\beta)}} 1^{\underline{p+1}}.$$

For this to give a contradiction we need

$$m' = n', \text{i.e. } \cancel{\alpha} + k\beta + p - (\cancel{\alpha} + \beta) = \underline{p + (k-1)\beta} = p + 1$$

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3. But, we can't control β, so this w does not work

Let's try again!!!

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2. Consider what happens when we pump k times:

$$xy^kz = 0^{\alpha+k\beta+m-(\alpha+\beta)}1^n.$$

$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot p$

$= p!$

We need a k s.t. $m + (k-1)\beta = n$ for a contradiction
Equivalently, we need $k = 1 + (n-m)/\beta$ to be an integer

Want $n - m$ divisible by β

s.t. $0 < \beta \leq p$

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   Equivalently, we need $k = 1 + (n-m)/\beta$ to be an integer
3. We only know $\beta \leq p$, how can we guarantee $(n-m)$ is divisible by β?

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Hint: What number is divisible by all integers $\leq p$?

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2. Consider what happens when we pump k times:

   $w = 0^{p!} 1^{2p!}$

$$xy^kz = 0^{\alpha+k\beta+m-(\alpha+\beta)}1^n.$$

   We need a k s.t. $m + (k-1)\beta = n$ for a contradiction
   Equivalently, we need $k = 1 + (n-m)/\beta$ to be an integer
3. We only know $\beta \leq p$, how can we guarantee $(n-m)$ is divisible by β?

   Hint: What number is divisible by all integers $\leq p$?
4. Set $n = 2(p!)$, $m = p!$, then $(n-m) = p!$ is divisible by β, so there is k s.t. $xy^kz \notin L$

Hints for Using the Pumping Lemma

To use the pumping lemma, need to do the following

- Identify what it means for $x \notin L$
- Choose w such that any valid split xyz can lead to a contradiction
- Prove that $w' = xy^kz \notin L$ form some k

Choosing w is often tricky, requires intuition and some trial and error.

Outline

1. Lecture 6 Review
2. Proving Languages Not Regular
3. Using the Pumping Lemma
4. **Using Closure Properties**
5. Pushdown Automata

Proving Non-Regularity Using Closure Properties

Consider $L = \{w | w \text{ has an equal number of 0s and 1s}\}$, prove L is not regular

A simpler proof:

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A simpler proof:

1. We already proved that $L_1 = \{0^n 1^n | n \geq 0\}$ is nonregular

Proving Non-Regularity Using Closure Properties

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A simpler proof:

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2. Observe that $L_1 = L \cap 0^*1^*$

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A simpler proof:

1. We already proved that $L_1 = \{0^n1^n|n \geq 0\}$ is nonregular
2. Observe that $L_1 = L \cap 0^*1^*$
3. Easy to see that 0^*1^* is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular

Proving Non-Regularity Using Closure Properties

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A simpler proof:

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2. Observe that $L_1 = L \cap 0^* 1^*$
3. Easy to see that $0^* 1^*$ is regular
4. Since regular languages are closed under $\cap$, if L is regular then L_1 must be regular
5. Since we know L_1 is nonregular, this means that L must be nonregular

Using Closure Properties of Regular Languages

We have seen a number of closure properties of REs

1. Closure under complement: $\overline{L}$ is regular if L is
2. Closure under union: $L_1 \cup L_2$ is regular if L_1, L_2 are
3. Closure under intersection: $L_1 \cap L_2$ is regular if L_1, L_2 are
4. Closure under reversal: L^R is regular if L is
5. NOPREFIX, NOEXTEND
6. There are many more (e.g., set difference, cross product, ...)

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Important

- It is often much easier to prove non-regularity using closure properties
- Try this first before you turn to pumping lemma

Exercise

Prove that the following language is nonregular:

$$L = \{0^i 1^j 2^i 3^j | i, j > 0\}$$

$w = 0^p 1^1 2^p 3^1$

$|xy| \le p$

$y = 0^p$